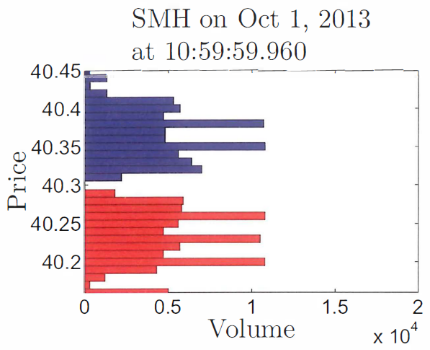
Memory in Quantitative Finance: Volterra Processes and Path-Signatures

Chapter 3 – Trading with a Volterra market impact

Eduardo Abi Jaber

- A trader aims to liquidate an initial position X_0 in shares over a time period $[0, T]$.
- During this liquidation, they will face **market impact**, meaning the asset price will drop as they sell.
- To minimize this impact, the trader will need to split the order into smaller trades over time.
- This creates a **trade-off** between **price impact** (selling quickly, but at a lower price) and **price risk** (selling slower, but risking adverse price movements).
- The trader will also consider available **trading signals** to optimize their strategy.

Q What type of market impact?
A temporary/instantaneous and transient/permanent



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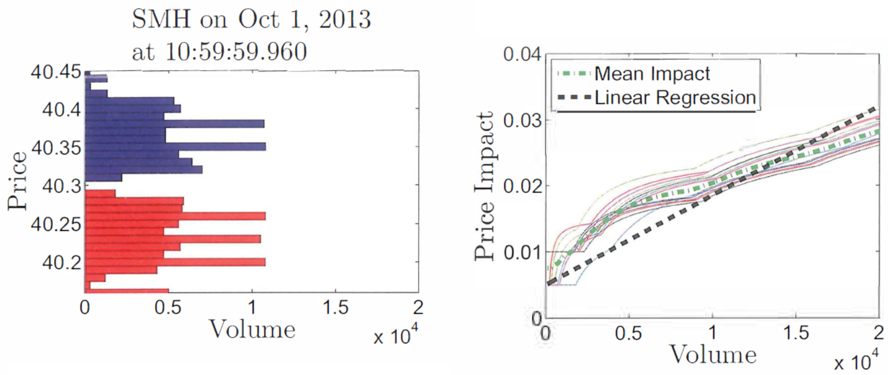

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Instantaneous/Transient impact justified by

- Equilibrium models such as Kyle (85): the optimal strategy for an investor with insider information on the fundamental price of an asset is to trade incrementally through time: to minimize costs and revelation of information to the rest of the market.
- Empirical studies indicate a persistence of power-law behavior in time of Market Impact and cross-market impact: Mastromatteo, Benzaquen, Eisler, & Bouchaud (2017) Bouchaud, Bonart, Donier & Gould (2018) Capponi & Cont (2018) . . .

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Rich literature on optimal trading with instanteneous/transient impact

- Gatheral, Schied and Shynkp (2012) Gârleanu & Pedersen (2013) Obizhaeva & Wang (2013) Bank, Soner & Voss (2017) Alfonsi, Klöck & Schied (2017) Ekren & Muhle Karbe (2019) Lehalle & Neuman (2019) Horst & Xia (2019) Becherer & Billarev (2024)...

Stylized model: single trader and multiple assets with cross-impact

Ref

- *Optimal Portfolio Choice with Cross-Impact Propagators*, with Eyal Neuman and Sturmius Tuschman (2024)
- *Optimal liquidation with signals: the general propagator case*, with Eyal Neuman (2022).

We consider a trader with an initial position of $X_0 \in \mathbb{R}^N$ shares in N-risky assets. The number of shares the trader holds at time $t \in [0, T]$ is prescribed as

$$X_t^u = X_0 + \int_0^t u_s ds,$$

where $(u_s)_{s\in[0,T]}$ denotes the trading speed.
Then we introduce the actual price in which the orders are executed by:

$$S_t := \underbrace{P_t}_{\text{signal inclusion}} + \frac{1}{2}\underbrace{\Lambda u_t}_{\text{instanteneous}} + \underbrace{D_t^u}_{\text{transient}}, \qquad 0 \le t \le T,$$

where P denotes some unaffected price process with covariance matrix Σ and

$$D_t^u = \int_0^t G(t,s) u_s ds, \qquad 0 \le t \le T,$$

for some square integrable transient impact kernel $G : [0, T] \to \mathbb{R}^{N\times N}$.

Large class of transient impact kernels that offer great modeling flexibility (factorized forms of Benzaquen, Mastromatteo, Eisler, & Bouchaud (2017)):

- Permanent impact

$$G(t,s) = 1_{s<t}A$$

- Exponential kernel

$$G(t,s) = e^{-\lambda(t-s)}1_{s<t}A, \quad \lambda \in \mathbb{R}$$

- Fractional kernel

$$G(t,s) = (t-s)^{H-1/2}1_{s<t}A, \quad H \in (0,1),$$

- Non-convolution for bond with maturity T trading

$$G(t,s) = f(t-T)H(t-s)A$$

  with $f(0) = 0$ due to the terminal condition on the bond price.

and A is a $N \times N$-matrix.

Maximize objective functional:

$$J(u) = \mathbb{E}\Big[\int_0^T -u_t^\top (P_t + D_t^u + \frac{1}{2}\Lambda u_t)dt + (X_T^u)^\top P_T - \frac{\gamma}{2}\int_0^T (X_t^u)^\top \Sigma X_t^u dt - \frac{\varrho}{2}(X_T^u)^\top \Pi X_T^u\Big]$$

- The first two terms represent the trader's terminal wealth: final cash position including the accrued trading costs which are induced by temporary and transient price impact, as well as remaining final risky asset position's book value:

$$-\int_0^T S_t dX_t^u \approx -\sum_i S_{t_i}(X_{t_{i+1}}^u - X_{t_i}^u)$$

- The third and fourth terms implement a risk aversion term $\gamma > 0$ and a penalty $\varrho > 0$ on running and terminal inventory.

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- Admissible set

$$\mathcal{U} := \Big\{u : \Omega \times [0, T] \to \mathbb{R}^N \Big| u \text{ progressively measurable with } \mathbb{E}\Big[\int_0^T \|u_t\|^2 dt\Big] < \infty\Big\}.$$

- Admissible kernels: Volterra $(G(t,s) = 0$ for $s > t)$ square integrable and nonnegative definite kernels G, if it holds for every $f \in L^2([0,T], \mathbb{R}^N)$ that

$$\int_0^T \int_0^T f(t)^\top G(t,s) f(s) ds dt \geq 0.$$

- For such kernels, the map $u \mapsto J(u)$ is strictly concave in $u \in \mathcal{U}$.

By strict concavity, the objective functional J admits a unique maximizer characterized by the critical point $u \in \mathcal{U}$ at which its Gâteaux derivative

$$\langle J'(u), v \rangle := \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon}$$

vanishes for any $v \in \mathcal{U}$.

Gâteaux derivative

For any $u, v \in \mathcal{U}$, the Gâteaux derivative of J is given by

$$\langle J'(u), v \rangle = \mathbb{E}\left[\int_0^T v_t^\top \Big(-P_t - ((\mathbf{G} + \mathbf{G}^*)u)(t) - \Lambda u_t + P_T - \gamma \int_t^T \Sigma X_s^u ds \Big) dt \right].$$

Notation:

$$(\mathbf{G}u)(t) := \int_0^t G(t,s) u_s ds \quad (\mathbf{G}^* u)(t) = \int_t^T G(s,t) u_s ds.$$

FOC equation

The unique maximizer of the objective functional $J(u)$ satisfies the linear equation

$$\Lambda u_t = f_t - \int_0^t K(t,s) u_s ds - \int_t^T K(s,t)^\top \mathbb{E}_t[u_s] ds, \quad 0 \le t \le T,$$

with

$$f_t := \mathbb{E}_t[P_T - P_t] - (\gamma(T-t)\Sigma + \varrho\Pi)X_0,$$

$$K(t,s) := G(t,s) + \mathbb{1}_{\{t>s\}}((T-t)\gamma\Sigma + \varrho\Pi)$$

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This is a non-standard stochastic (adaptive) Fredholm equation with both forward and backward components.

Back to 1903

- **Ivar Fredholm** (1903). Sur une classe d'équations fonctionnelles. Acta mathematica, 27(1), 365-390.
- His work foreshadowed the theory of **David Hilbert** on Hilbert spaces.
- **Henry McKean** (2010): "Nowadays, Fredholm's determinant is explained mostly in an abstract and, if I may say so, a **superficial way**."
- **Peter Lax** (2002) book: entire chapter for Fredholm's theory: "it is time to **resurrect** Fredholm's determinant."

Fredholm determinant

From Wikipedia, the free encyclopedia

In mathematics, the **Fredholm determinant** is a complex-valued function which generalizes the determinant of a finite dimensional linear operator. It is defined for bounded operators on a Hilbert space which differ from the identity operator by a trace-class operator. The function is named after the mathematician Erik Ivar Fredholm.

Definition [edit]

Let H be a Hilbert space and G the set of bounded invertible operators on H of the form $I + T$, where T is a trace-class operator. G is a group because

$$(I+T)^{-1} - I = -T(I+T)^{-1},$$

so $(I+T)^{-1}-I$ is trace class if T is. It has a natural metric given by $d(X, Y) = \|X - Y\|_1$, where $\|\cdot\|_1$ is the trace-class norm.

If H is a Hilbert space with inner product $(\cdot, \cdot)$, then so too is the kth exterior power $\Lambda^k H$ with inner product

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k) = \det(v_i, w_j).$$

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Déterminant de Fredholm

4 langues

Article Discussion Plus

$$\phi(x) + \int_0^1 K(x,t)\phi(t)\,dt = f(x) \quad (*).$$

On peut essayer de discrétiser cette équation :

- en l'évaluant sur une famille de points $(x_l)_{0\leq l\leq n-1}$ équirépartis dans l'intervalle $[0;1]$: $x_k = \frac{k}{n},\ 1 \leq l \leq n-1.$
- en approchant l'intégrale par une somme de Riemann : $\int_0^1 k(x_i,t)\phi(t)\,dt \approx \sum_{j=0}^{n-1} \frac{1}{n} k(x_i,x_j)\phi(x_j).$

On obtient alors, pour chaque $n \in \mathbb{N}$, un système linéaire (E_n) d'équations

$$(E_n) : \begin{cases} \phi(x_1) + \frac{1}{n}\sum_{j=0}^{n-1} k(x_1,x_j)\phi(x_j) = f(x_1) \\ \dots \\ \phi(x_i) + \frac{1}{n}\sum_{j=0}^{n-1} k(x_i,x_j)\phi(x_j) = f(x_i) \\ \dots \\ \phi(x_n) + \frac{1}{n}\sum_{j=0}^{n-1} k(x_n,x_j)\phi(x_j) = f(x_n) \end{cases}$$

- Deterministic linear Fredholm equation (1903):

$$v(t) = a(t) + \int_0^T B(t,s)v(s)ds$$

modulo invertibility, unique solution:

$$v(t) = ((\mathrm{id} - \boldsymbol{B})^{-1} a)(t)$$

- Invertibility readily obtained for:
  - B is a Volterra kernel $B(t,s) = 0$ for $s \geq t$ $((B(t_i, t_j))_{i,j \leq n}$ 'lower triangular matrix')
  - B symmetric negative semidefinite $\int_0^T \int_0^T f(t)B(t,s)f(s)dtds \geq 0$, for all f.

FOC is equivalent to:

$$u_t = \underbrace{f_t}_{\text{stochastic}} - \underbrace{\int_0^t K(t,r)u_r dr}_{\text{forward}} - \underbrace{\int_t^T L(r,t)\mathbb{E}_t u_r dr}_{\text{backward}}, \quad t \in [0,T],$$

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Intuition Express $\mathbb{E}_t u_r$ as a linear functional of $(u_r)_{r\leq t}$.

Two step approach

1. For fixed r define the process

$$m_r(s) = 1_{r<s}\mathbb{E}_r[u_s]$$

Taking conditional expectation and tower property yields a (standard) Fredholm equation for $m_r(s)$ in terms of $(u)_{s\leq r}$

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Two step approach

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   Taking conditional expectation and tower property yields a (standard) Fredholm equation for $m_r(s)$ in terms of $(u)_{s \leq r}$
2. Solve for m_r and plug-it in the initial equation and solve the new standard Fredholm equation for u.

Notation: define $G_t(s,r) = G(s,r)\mathbb{1}_{r\geq t}$, and we denote by $\boldsymbol{G}_t$ the associated Volterra operator.

Theorem: Adaptive Fredholm

Let $K, L : [0,T]^2 \to \mathbb{R}$ be two Volterra kernels such that the operator $\boldsymbol{D}_t = \mathrm{id} + \boldsymbol{K}_t + \boldsymbol{L}_t^*$ is invertible, $(f_t)_{t\leq T}$ be progressive s.t. $\int_0^T \mathbb{E}[f_t^2]dt < \infty$. Then, the linear adaptive Fredholm eq

$$v_t = f_t - \int_0^t K(t,r)v_r dr - \int_t^T L(r,t)\mathbb{E}_t v_r dr, \quad t \in [0,T],$$

admits a unique progressively measurable solution $(v_t)_{0\leq t\leq T}$ given by

$$v_t = \left((\mathrm{id} - \boldsymbol{B})^{-1} a\right)(t), \quad 0 \leq t \leq T,$$

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$$v_t = \left((\mathrm{id} - \boldsymbol{B})^{-1} a \right)(t), \quad 0 \leq t \leq T,$$

where

$$a_t = f_t - \langle 1_{t \leq \cdot} L(\cdot, t), \boldsymbol{D}_t^{-1} 1_{t \leq \cdot} \mathbb{E}_t f_\cdot \rangle_{L^2}$$

$$B(t,s) = 1_{\{s<t\}} \left(\left\langle 1_{t \leq \cdot} L(\cdot, t), \boldsymbol{D}_t^{-1} 1_{t \leq \cdot} K(\cdot, s) \right\rangle_{L^2} - K(t,s) \right),$$

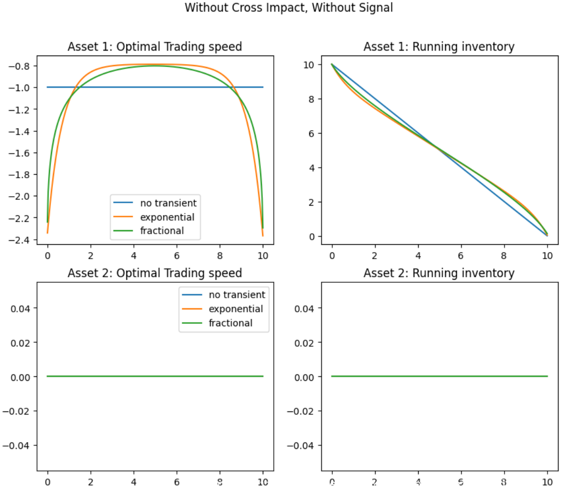
where $\boldsymbol{B}$ is the integral operator induced by the kernel B.

For the full proof and details see Section 5 in Abi Jaber, E. A., Neuman, E., & Voß, M. (2023). Equilibrium in functional stochastic games with mean-field interaction. arXiv preprint arXiv:2306.05433.

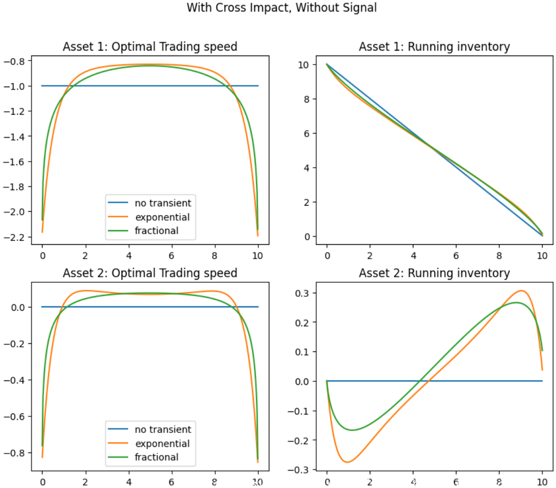
Propagator model with cross-impact

Numerics

Without cross-impact $A = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.06 \end{pmatrix}$



With cross-impact $A = \begin{pmatrix} 0.06 & 0.05 \\ 0.05 & 0.06 \end{pmatrix}$



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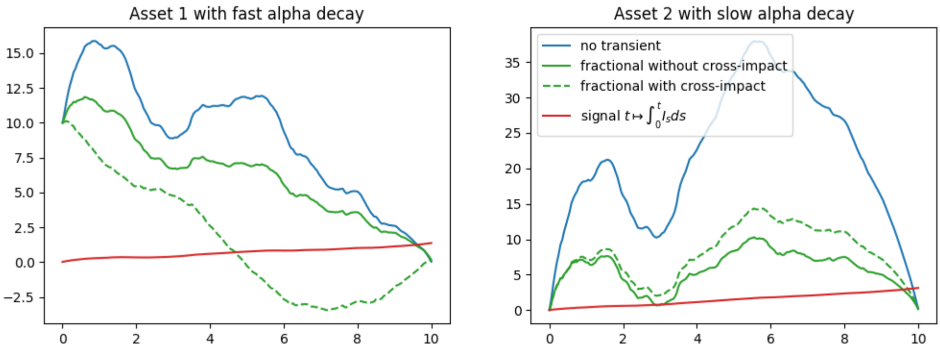

Buy trading signals with different alpha decay on each asset given by $(\beta_1, \beta_2) = (0.9, 0.3)$ **without (solid) and with (dashed) cross-impact**.

Propagator model with cross-impact

4 stylized models

- single trader and multiple assets: Cross-impact
- single trader with trading constraints: e.g. No shorting or buying
  Abi Jaber, E. De Carvalho, N., & Pham, H. (2024). Trading with propagators and constraints: applications to optimal execution and battery storage. arXiv preprint arXiv:2409.12098.
- multiple traders and single asset: aggregated transient impact
  Abi Jaber, E., Neuman, E., & Voß, M. (2023). Equilibrium in functional stochastic games with mean-field interaction. arXiv preprint arXiv:2306.05433.
- single trader and non-linear market impact
  Abi Jaber, E., Bondi, A., De Carvalho, N., Neuman, E., & Tuschmann, S. (2025). Fredholm Approach to Nonlinear Propagator Models. Available at SSRN 5167824.

- Regulatory constraints, No shorting: $X_t^u \geq 0$
- No buying in the presence of buying signal: $u \leq 0$
- Full liquidation constraint: $X_T^u = 0$.
- Physical constraints in battery storage problem: charging/discharging speed u and capacity $X^{min} \leq X^u \leq X^{max}$

Same objective functional without penalty terms

$$J(u) = \mathbb{E}\Big[\int_0^T -u_t(P_t + D_t^u + \frac{1}{2}\lambda u_t)dt + (X_T^u)P_T\Big], \quad D_t^u = \int_0^t G(t,s)u_s ds$$

but optimized on the admissible set

$$\mathcal{U} := \left\{u \in \mathcal{L}^\infty, \;\; a_t^1 \le u_t \le a_t^2 \text{ and } a_t^3 \le X_t^u \le a_t^4 \text{ hold } \; (dt \otimes \mathbb{P}) - a.e. \text{ on } [0,T] \times \Omega\right\}.$$

with $a_1, a_2, a_3, a_4 \in \mathcal{L}^\infty$.

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with $a_1, a_2, a_3, a_4 \in \mathcal{L}^\infty$.

Existence and uniqueness

Assume $\mathcal{U}$ is non-empty and let G be an admissible kernel and $\lambda > 0$. There exists a unique optimal admissible control $u^* \in \mathcal{U}$ such that $\mathcal{J}(u^*) = \sup_{u\in\mathcal{U}} \mathcal{J}(u)$.

Introduce the constraining operator $H : \mathcal{L}^\infty \to (\mathcal{L}^\infty)^4$:

$$H(u) := \begin{pmatrix} a^1 - u \\ u - a^2 \\ a_t^3 - X_t^u \\ X_t^u - a_t^4 \end{pmatrix}.$$

Define the Lagrangian

$$L(u, \psi) := J(u) + \langle H(u), \psi \rangle_{(\mathcal{L}^\infty)^4, ((\mathcal{L}^\infty)^4)^\star}, \quad u \in \mathcal{L}^\infty, \quad \psi \in \boldsymbol{ba}_+^4.$$

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- $\boldsymbol{ba}$ is the dual space of $\mathcal{L}^\infty$: space of bounded finitely additive set-valued functions on $\mathcal{B}([0,T]) \otimes \mathcal{F}$ which vanish on sets of $dt \otimes P$-measure zero.
- working on $\mathcal{L}^\infty$ is needed to obtain existence of ψ using standard separation theorems, which require non-empty interiors of positive cones. (L^p_+ has empty interior in L^p for any $p < \infty$.)

We expect the optimal control u^* to satisfy:

(i) the Lagrange stationary equation in the form

$$\underbrace{\nabla_u J(u^*)}_{\text{Fredholm}} + \langle \underbrace{\nabla_u H(u^*)}_{\text{linear constraints}}, \psi\rangle = 0_{(\mathcal{L}^\infty)^4}$$

where ∇_u are Gateaux derivatives,

(ii) complementary slackness conditions given by

$$\begin{cases} 0 = \int_{[0,T]\times\Omega} \left(a_t^1(\omega) - \hat{u}_t(\omega)\right) \psi^1(dt, d\omega) \\ 0 = \int_{[0,T]\times\Omega} \left(\hat{u}_t(\omega) - a_t^2(\omega)\right) \psi^2(dt, d\omega) \\ 0 = \int_{[0,T]\times\Omega} \left(a_t^3(\omega) - X_t^{\hat{u}}(\omega)\right) \psi^3(dt, d\omega) \\ 0 = \int_{[0,T]\times\Omega} \left(X_t^{\hat{u}}(\omega) - a_t^4(\omega)\right) \psi^4(dt, d\omega) \end{cases}.$$

Main result

Assume existence of u satisfying the strict inequalities, $\lambda > 0$ and let G be an admissible kernel. Then, there exist Lagrange multipliers $(\psi^1, \psi^2, \psi^3, \psi^4) \in (\boldsymbol{ba}_+)^4$ satisfying the decomposition

$$\psi^1(t, d\omega) - \psi^2(t, d\omega) + \mathbb{E}_t\psi^3([t, T], d\omega)dt - \mathbb{E}_t\psi^4([t, T], d\omega)dt = \mu_t(\omega)dt\mathbb{P}(d\omega),$$

with $\mu \in \mathcal{L}^2$, as well as the complementary slackness with the process u^* solving the stochastic Fredholm equation of the second kind of the form

$$u^*_t = f_t + \mu_t - \int_0^t G(t, s)u^*_s ds - \int_t^T G(s, t)\mathbb{E}_t u^*_s ds,$$

Furthermore, such u^* is the unique admissible control satisfying

$$J(u^*) = \inf_{u \in \mathcal{U}} J(u).$$

- If we know μ_t and $\mathbb{E}_t\mu_s$ we can solve u^* explicitly.

Uzawa meets Longstaff-Schwarz

- Consider the dual problem

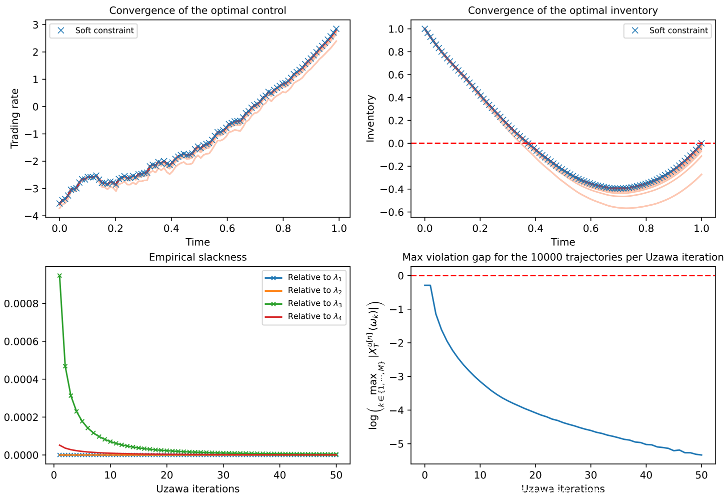
$$\inf_{\psi \geq 0} \sup_{u} L(u, \psi)$$

  for a given ψ, inner problem reduces to solving stochastic Fredholm equation.
- Uzawa algorithm: projected gradient ascent to obtain ψ

$$\begin{cases} \psi[0] & = & 0, \\ u^*[n] & = & \arg\max_u L^N(u, \psi[n]), \\ \psi[n+1] & = & (\psi[n] + \delta H(u^*[n]))^+ \end{cases}$$

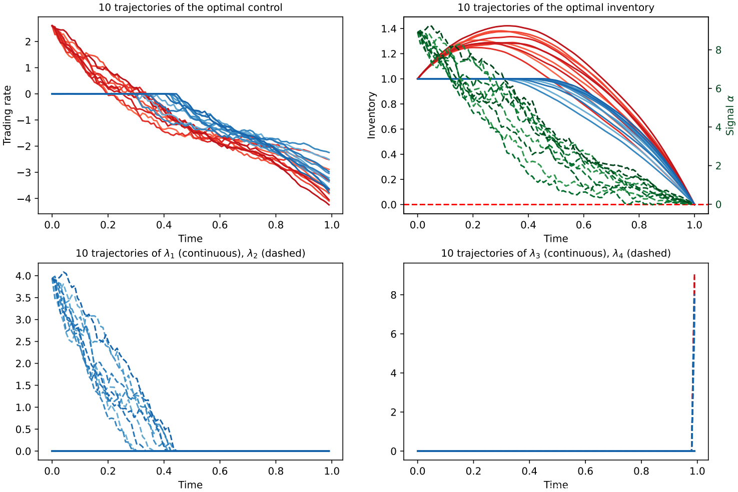
- Except that stochastic Fredholm equations require the knowledge of $\mathbb{E}_t[\psi] \Rightarrow$ Least-Squares MC to learn these on specific basis in Markovian settings ie. G is a sum of exponential and signal is Markovian.

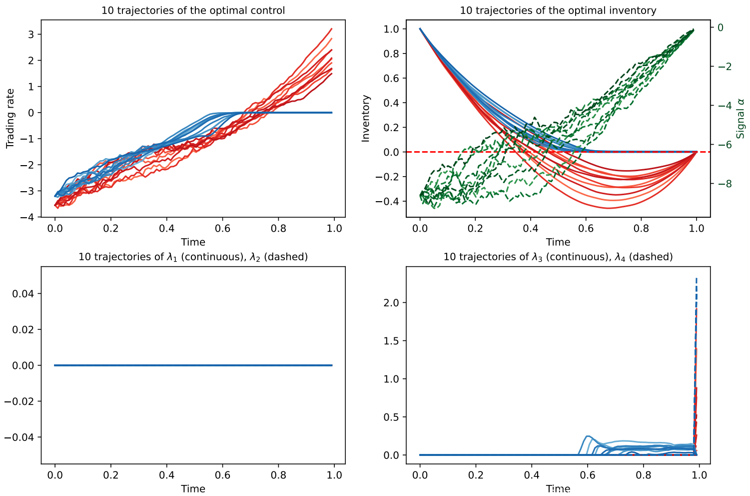
Propagator model with constraints
Numerics: Sanity check



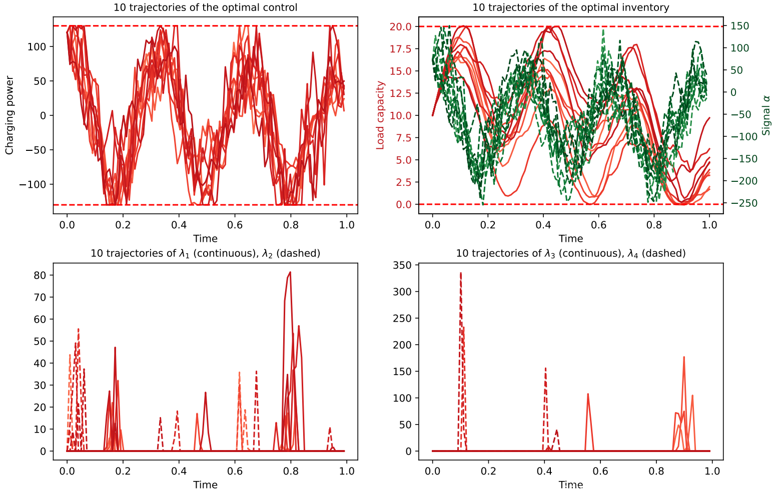
Propagator model with constraints

Numerics: No buying





10 trajectories of the optimal control
Trading rate
Time
10 trajectories of the optimal inventory
Inventory
Signal α
Time
10 trajectories of λ_1 (continuous), λ_2 (dashed)
Time
10 trajectories of λ_3 (continuous), λ_4 (dashed)
Time


10 trajectories of the optimal control
Charging power
Time
10 trajectories of the optimal inventory
Load capacity
Signal α
Time
10 trajectories of λ_1 (continuous), λ_2 (dashed)
Time
10 trajectories of λ_3 (continuous), λ_4 (dashed)
Time

N **traders** with initial position $X_0^i \in \mathbb{R}$ shares in a risky asset for i-th trader and

$$X_t^i = X_0^i + \int_0^t u_s^i ds,$$

where $(u_s^i)_{s\in[0,T]}$ denotes the selling speed which is chosen by the trader. Traders' individual and aggregated trading activity causes price impact on the risky asset price:

$$S_t^i := \underbrace{P_t^i}_{\text{Signal inclusion}} + \underbrace{\lambda u_t^i}_{\text{instanteneous}} + \underbrace{D_t^{\bar{u}}}_{\text{transient}},$$

where N^i denotes some unaffected progressively measurable price process incorporating an individual (or common) trading signal; the process

$$D_t^{\bar{u}} = \int_0^t G(t,s)\bar{u}_s ds, \quad \bar{u}_t := \frac{1}{N}\sum_{i=1}^N u_t^i,$$

captures aggregated linear decaying price impact with general propagator kernel G; $\lambda > 0$ instantaneous slippage costs.

Objective criterion each trader maximize $P\&L$ which leads to

$$J^{i,N}(u^i) := \mathbb{E}\left[\int_0^T (P_t^i - D_t^{\bar{u}})u_t^i dt - \lambda \int_0^T (u_t^i)^2 dt + X_T^i P_T - \phi \int_0^T (X_t^i)^2 dt - \varrho (X_T^i)^2\right],$$

Nash equilibrium

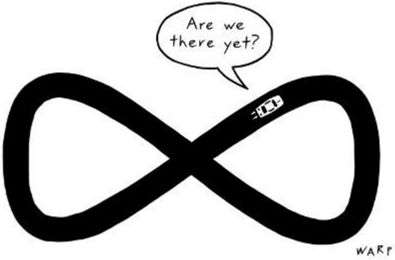
A set of strategies $\hat{u}^N = (\hat{u}^{1,N}, \ldots, \hat{u}^{N,N}) \in \mathcal{U}^N$ is called a Nash equilibrium if

$$J^{i,N}(\hat{u}^{i,N}; \hat{u}^{-i,N}) \geq J^{i,N}(v; \hat{u}^{-i,N}), \quad v \in \mathcal{U}, \quad i \in \{1, \ldots, N\}.$$

Mean Field problem

$$J^i(u^i) = \mathbb{E}\left[\int_0^T (P_t^i - D_t^{\bar{u}})u_t^i dt - \lambda \int_0^T (u_t^i)^2 dt + X_T^i P_T - \phi \int_0^T (X_t^i)^2 dt - \varrho (X_T^i)^2\right],$$

$$i = 1, \ldots, N, \quad N \to \infty$$



N-player game

$$J^i(u^i) := \mathbb{E}\left[\int_0^T (P_t^i - D_t^{\bar{u}})u_t^i dt - \lambda \int_0^T (u_t^i)^2 dt + X_T^i P_T - \phi \int_0^T (X_t^i)^2 dt - \varrho (X_T^i)^2\right],$$

$$i = 1, \ldots, N, \qquad \bar{u}_t = \frac{1}{N}\sum_{j=1}^N u_t^j$$

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$$i = 1, \dots, N, \qquad \bar{u}_t = \frac{1}{N}\sum_{j=1}^N u_t^j$$

To motivate assume:

$$P^i = \underbrace{\beta^i}_{\text{individual}} + \underbrace{\beta^0}_{\text{common}} \qquad \text{with } (\beta^i)_{i=1,\dots,N} \text{ independent.}$$

Propagator model with interactions
Mean Field problem

N-player game

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'$N \to \infty$': One representative player with strategy v with signal

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'$N \to \infty$': One representative player with strategy v with signal

$$P = \underbrace{\beta}_{\textit{individual}} + \underbrace{\beta^0}_{\textit{common}}$$

$$J^\infty(v) = \mathbb{E}\left[\int_0^T (P_t - D_t^{\mu})v_t dt - \lambda \int_0^T (v_t)^2 dt + X_T^v P_T - \phi \int_0^T (X_t^v)^2 dt - \varrho (X_T^v)^2\right] \qquad \mu_t = \mathbb{E}[v_t]?$$

Propagator model with interactions
Mean Field problem

N-player game

$$J^i(u^i) := \mathbb{E}\left[\int_0^T (P_t^i - D_t^{\bar{u}})u_t^i dt - \lambda \int_0^T (u_t^i)^2 dt + X_T^i P_T - \phi \int_0^T (X_t^i)^2 dt - \varrho (X_T^i)^2\right],$$

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$$
\begin{aligned}
J(v,\mu) =& \mathbb{E}\left[\int_0^T (P_t - D_t^{\mu}) v_t dt - \lambda \int_0^T (v_t)^2 dt + X_T^v P_T - \phi \int_0^T (X_t^v)^2 dt - \varrho (X_T^v)^2\right] \\
P_t =& \beta_t + \beta_t^0
\end{aligned}
$$

Definition

A pair $(\hat{v}, \hat{\mu}) \in \mathcal{U}_{\mathbb{F}^b} \times \mathcal{U}_{\mathbb{F}^{\beta^0}}$ is called a mean-field Nash equilibrium if $\hat{v}$ solves

$$J(\hat{v}; \hat{\mu}) \geq J(u; \hat{\mu}), \quad u \in \mathcal{U}_{\mathbb{F}^b},$$

with the consistency condition

$$\mathbb{E}\left[\hat{v}_t | \mathcal{F}_T^{\beta^0}\right] = \hat{\mu}_t \quad \Omega \times [0, T] \text{ almost everywhere.}$$

Set

$$f_t := \mathbb{E}_t[P_T - P_t] - (2(T-t)\phi + 2\varrho)X_0,$$

$$K(t,s) := G(t,s) + \mathbb{1}_{\{t>s\}}((T-t)2\phi + 2\varrho)$$

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Fix μ. Maximize $v \to J(v,\mu)$. FOC lead again to a stochastic Fredholm equation for $\hat{v}$ in the form:

$$2\lambda \hat{v}_s = f_s - \int_0^s G(s,r)\mu_r dr - \int_0^s K(s,r)\hat{v}_r dr - \int_s^T K(r,s)\mathbb{E}_s \hat{v}_r dr.$$

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and by conditioning this equation on $\mathcal{F}_T^{\beta^0}$ a candidate equation for $\hat{\mu}$ given by

$$2\lambda\hat{\mu}_s = \mathbb{E}[f_s|\mathcal{F}_T^{\beta^0}] - \int_0^s (K(s,r) + G(s,r))\,\hat{\mu}_r dr - \int_s^T K(r,s)\mathbb{E}_s[\hat{\mu}_r]dr.$$

The solution of the mean field game will involve the following solution map F of the corresponding Fredholm equation:

$$F(t,x) = \left((\mathrm{id} - \tilde{\boldsymbol{B}})^{-1} \tilde{a}^x \right)(t) \qquad (0 \le t \le T)$$

where

$$\tilde{a}^x_t = \frac{1}{2\lambda} \left(x_t - \langle 1_{t\le \cdot} K(\cdot, t), \boldsymbol{D}_t^{-1} 1_{t \le \cdot} \mathbb{E}_t[x_\cdot] \rangle_{L^2} \right),$$

$$\tilde{B}(t,s) := 1_{\{s \le t\}} \frac{1}{2\lambda} \left(\langle 1_{t\le \cdot} K(\cdot, t), \boldsymbol{D}_t^{-1} 1_{t\le \cdot} K(\cdot, s) \rangle_{L^2} - K(t,s) \right),$$

$$\boldsymbol{D}_t := 2\lambda \mathrm{id} + (\boldsymbol{K})_t + (\boldsymbol{K}^*)_t.$$

Theorem: MFG solution

Assume that the linear operators $\boldsymbol{D}_t$ and $\hat{\boldsymbol{D}}_t := 2\lambda \mathrm{id} + (\boldsymbol{G})_t + (\boldsymbol{K})_t + (\boldsymbol{K}_2^*)_t$ are invertible for all $t \leq T$. Then, the mean field Nash equilibrium strategy $(\hat{v}, \hat{\mu})$ is given by

$$\hat{v}_t = F(t, f - \boldsymbol{G}\hat{\mu}), \quad \hat{\mu}_t = \left((\mathrm{id} - \hat{\boldsymbol{B}})^{-1}\hat{a}\right)(t),$$

where $\hat{a}_t := \frac{1}{2\lambda}\left(\beta_t^0 + \mathbb{E}[\beta_t] - \langle 1_{t\leq \cdot} K(\cdot, t), \hat{\boldsymbol{D}}_t^{-1} 1_{t\leq \cdot}\, (\mathbb{E}_t[\beta^0_\cdot] + \mathbb{E}[\beta_\cdot])\rangle_{L^2}\right),$

$$\hat{B}(t,s) := 1_{\{s\leq t\}} \frac{1}{2\lambda}\left(\langle 1_{t\leq \cdot} K(\cdot, t), \hat{\boldsymbol{D}}_t^{-1} 1_{t\leq \cdot}(G(\cdot, s) + K(\cdot, s))\rangle_{L^2} - (G(t,s) + K(t,s))\right),$$

for all $t \in [0, T]$, where F is the solution map of previous slide.

Back to finite player game

$$J^i(u^i) := \mathbb{E}\left[\int_0^T (P_t^i - D_t^{\bar{u}})u_t^i dt - \lambda \int_0^T (u_t^i)^2 dt + X_T^i P_T - \phi \int_0^T (X_t^i)^2 dt - \varrho (X_T^i)^2\right],$$

$$i = 1, \dots, N, \qquad \bar{u}_t = \frac{1}{N}\sum_{j=1}^N u_t^j$$

Can be solved using similar strategy and has similar structure:

Nash equilibrium

$$\hat{u}^i_t = ((\mathrm{id} - \mathrm{B}^N)^{-1} a^i), \quad i = 1, \ldots, N$$

for explicit operator B^N and process a^i.

1. Stability of adaptive Fredholm equation:
   - convergence of the N-player game towards the game with a infinite number of agents as $N \to \infty$
   - ϵ-Nash equilibrium from MFG game using the Fredholm map F of the mean field solution but plugging individual signals P^i:

$$\hat{v}^i_t = \boldsymbol{F}(t, f^i - \boldsymbol{G}\hat{\mu})$$

2. Rates of convergence

Final remark on LQ structure

Link with Riccati equations The solution depends on the operator:

$$\boldsymbol{D}_t := \mathrm{id} + \boldsymbol{K}_t + \boldsymbol{L}_t^* = \mathrm{id} + \boldsymbol{H}_t$$

more precisely on the inverse $\boldsymbol{\Psi}_t = \boldsymbol{D}_t^{-1}$. Differentiating in time

$$\dot{\boldsymbol{\Psi}}_t = -\boldsymbol{D}_t^{-1}\dot{\boldsymbol{D}}_t\boldsymbol{D}_t^{-1} = -\boldsymbol{\Psi}_t\dot{\boldsymbol{H}}_t\boldsymbol{\Psi}_t$$

which is an infinite dimensional Riccati equations, reminescent of Riccati equations that arise in (finite and infinite dimensional) Markovian Linear Quadratic control problems

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Value function Direct computations show that value function V^* is **linear quadratic** in L^2 space wrt suitable forward process $g_t^u(s) := \mathbb{E}_t[X_s^u]$ (where X^u are the controlled variables, eg $X_t^u = \int_0^t G(t,s)u_s ds + B_t$) in the sense that

$$V_t^* = \langle g_t^{u^*}, \boldsymbol{\Psi}_t g_t^{u^*} \rangle_{L^2} + \langle \Theta, g_t^{u^*} \rangle_{L^2} + \phi_t$$

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Optimal control is in **linear feedback form**

$$u_t^* = a_t + (\boldsymbol{\Gamma}_t g_t^{u^*})(s)$$

for some linear operator $\boldsymbol{\Gamma}_t$.

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