Chapter 9

Viscosity solutions for the dynamic programming equation

9.1 Viscosity solutions for parabolic PDEs

Let F be an operator from $[0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ into $\mathbb{R}$ where $\mathbb{S}^n$ stands for the set of n-dimensional symmetric matrices. In this chapter, we will be mostly interested by the case

$$F(t,x,u,q,p,M) \quad = \quad -\sup_{a\in A}\big\{\mathcal{L}^a[t,x,u,q,p,M] + f(x,a)\big\}$$

where

$$\mathcal{L}^a[t,x,u,q,p,M] \quad = \quad q + b(x,a).p + \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(x,a)M\big)$$

for $(t,x,u,q,p,M) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ where the coefficients b, σ and f and the set A are those of the previous chapters and satisfy the related assumptions. This case is called the HJB case.

In the sequel, we assume that F is an elliptic operator, that is, F is nonincreasing in the the variable M. We are interested in the PDE

$$F(t,x,u(t,x),\partial_t u(t,x),\nabla u(t,x),\nabla^2 u(t,x)) \quad = \quad 0 \tag{9.1.1}$$

for $(t,x) \in [0,T) \times \mathbb{R}^n$, together with the terminal condition

$$u(T,x) \quad = \quad g(x)\ , \quad x \in \mathbb{R}^n\ . \tag{9.1.2}$$

Our objective is to give a notion of weak solution to such a PDE.

Definition 9.1.1 *(i) A continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity subsolution to* (9.1.1) *if for any function $\varphi \in C^{1,2}[0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ such that*

$$0 = (u - \varphi)(t,x) = \max_{[0,T]\times\mathbb{R}^n} (u - \varphi)$$

we have

$$F(t, x, \varphi(t,x), \partial_t\varphi(t,x), \nabla\varphi(t,x), \nabla^2\varphi(t,x)) \leq 0 .$$

(ii) A continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity supersolution to (9.1.1) *if for any function $\varphi \in C^{1,2}[0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ such that*

$$0 = (u - \varphi)(t,x) = \min_{[0,T]\times\mathbb{R}^n} (u - \varphi)$$

we have

$$F(t, x, \varphi(t,x), \partial_t\varphi(t,x), \nabla\varphi(t,x), \nabla^2\varphi(t,x)) \geq 0 .$$

(iii) A function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity solution to (9.1.1) *if it is a viscosity subsolution to* (9.1.1) *and a viscosity supersolution to* (9.1.1).

9.2 Viscosity properties of the value function

We consider the HJB case for PDE (9.1.1) and the value function v defined by (6.1.3).

Theorem 9.2.1 *The value function v is a viscosity solution to* (9.1.1) *and satisfies* (9.1.2).

Proof. The fact that v satisfies (9.1.2) comes from its definition. We turn to the viscosity properties.

Step 1. Subsolution property. Let $(\hat{t}, \hat{x}) \in [0,T) \times \mathbb{R}^n$ and $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ be such that

$$0 = (v - \varphi)(\hat{t}, \hat{x}) = \max_{[0,T]\times\mathbb{R}^n} (v - \varphi) . \tag{9.2.1}$$

We proceed by contradiction and assume that

$$-\sup_{a\in A}\left\{\mathcal{L}^a\varphi(\hat t,\hat x)+f(\hat x,a)\right\} \;\;\geq\;\; 2\eta$$

for some $\eta>0$. By continuity of the coefficients, we can find $r\in(T-\hat t,T)$ such that

$$-\sup_{a\in A}\left\{\mathcal{L}^a\varphi(t,x)+f(x,a)\right\} \;\;\geq\;\; \eta \tag{9.2.2}$$

for $(t,x)\in B:=B(\hat t,r)\times B(\hat x,r)$. Without loss of generality, we can assume that the minimum in (9.2.3), is strict and that

$$\max_{\partial B}(v-\varphi) \;\;\leq\;\; -\eta\,.$$

Let ν be the first exit time of $(s,X_s^{t,x,\hat\alpha})_{s\geq t}$ from B. From the DPP, there exist a control $\hat\alpha\in\mathcal{A}_{\hat t}$ such that

$$v(\hat t,\hat x) \;\;\leq\;\; \mathbb{E}\Big[v(\theta,X_\theta^{t,x,\hat\alpha})+\int_t^\theta f\big(X_s^{t,x,\hat\alpha},\hat\alpha_s\big)ds\Big]+\frac{\eta}{2}\,.$$

From the previous inequalities, we get

$$\varphi(\hat t,\hat x) \;\;\leq\;\; \mathbb{E}\Big[\varphi(\theta,X_\theta^{t,x,\hat\alpha})-\eta+\int_t^\theta f\big(X_s^{t,x,\hat\alpha},\hat\alpha_s\big)ds\Big]+\frac{\eta}{2}\,.$$

Applying Itô's Lemma to φ, we get

$$0 \;\;\leq\;\; -\frac{\eta}{2}+\mathbb{E}\Big[\int_t^\theta\big(\mathcal{L}^{\alpha_s}\varphi\big(s,X_s^{t,x,\hat\alpha}\big)+f\big(X_s^{t,x,\hat\alpha},\hat\alpha_s\big)\big)ds\Big]\,.$$

Using (9.2.2) we get

$$0 \;\;\leq\;\; -\frac{\eta}{2}\,.$$

Step 2. Supersolution property. Let $(t,x)\in[0,T)\times\mathbb{R}^n$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ be such that

$$0 \;\;=\;\; (v-\varphi)(t,x) \;\;=\;\; \min_{[0,T]\times\mathbb{R}^n}(v-\varphi)\,. \tag{9.2.3}$$

Fix $a\in A$ and denote by $\hat\alpha$ the control constant equal to a. From the dynamic programming principle we have

$$v(t,x) \;\;\geq\;\; \mathbb{E}\Big[v\big(X_\theta^{t,x,\hat\alpha}\big)+\int_t^{t+h} f\big(X_s^{t,x,\hat\alpha},a\big)ds\Big]$$

for $h \in (0, T - \hat{t})$. Using (9.2.3), we get

$$\varphi(t,x) \;\; \geq \;\; \mathbb{E}\Big[\varphi\big(t+h, X_{t+h}^{t,x,\hat{\alpha}}\big) + \int_t^{t+h} f\big(X_s^{t,x,\hat{\alpha}}, a\big) ds\Big]$$

Applying Itô's formula and diving by $h > 0$, we get

$$0 \;\; \geq \;\; \frac{1}{h}\mathbb{E}\Big[\int_t^{t+h} \big[\mathcal{L}^a \varphi\big(X_s^{t,x,\hat{\alpha}}, a\big) + f\big(X_s^{t,x,\hat{\alpha}}, a\big)\big] ds\Big]$$

Sending h to $0+$, we get by the mean value theorem

$$\mathcal{L}^a \varphi\big(t,x\big) + f\big(x,a\big) \;\; \leq \;\; 0 \; .$$

Since this holds for all $a \in A$, we get the supersolution property.

9.3 Comparison and uniqueness

We first have the following result which allow to compare a supersolution and a subsolution as soon as we can compare their values at the terminal time T.

Theorem 9.3.1 *Let U (resp. V) $\in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ be a subsolution (resp. supersolution) with polynomial growth to* (9.1.1) *in the HJB case. If $U(T,.) \leq V(T,.)$ on $\mathbb{R}^n$, then $U \leq V$ on $[0,T] \times \mathbb{R}^n$.*

Proof. We present the proof in the regular case. The proof for viscosity solutions uses an additional result called Ishii's lemma which gives a similar condition to second order optimality condition. We proceed in 3 steps.
Step 1. Let $\tilde{U}(t,x) = e^{\lambda t} U(t,x)$ and $\tilde{V}(t,x) = e^{\lambda t} V(t,x)$. Then a straightforward calculation shows that $\tilde{U}$ (resp. $\tilde{V}$) is a subsolution (resp. supersolution) to

$$\tilde{F}\big(t, x, w, \partial_t w(t,x), \nabla w(t,x), \nabla w(t,x)\big) \;\; = \;\; 0 \tag{9.3.1}$$

with

$$\begin{aligned} F\big(t,x,u,q,p,M\big) \;\; &= \;\; -\sup_{a \in A} \big\{\tilde{\mathcal{L}}^a[t,x,u,q,p,M] + \tilde{f}(t,x,a)\big\} \; , \\ \tilde{\mathcal{L}}^a[t,x,u,q,p,M] \;\; &= \;\; q + b(x,a).p + \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(x,a)M\big) - \lambda u \; , \\ \tilde{f}(t,x,a) \;\; &= \;\; e^{\lambda t} f(x,a) \end{aligned}$$

for $(t,x,u,q,p,M) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ and $a \in A$.

Step 2. From the polynomial growth of U and V, we may choose an integer $p \geq 1$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \frac{|\tilde{U}(t,x)| + |\tilde{V}(t,x)|}{1+|x|^p} \quad < \quad +\infty \ ,$$

and we consider the function $\phi(t,x) = e^{-\lambda t}(1+|x|^{2p}) =: e^{-\lambda t}\psi(x)$. From the linear growth condition on b and σ, a straightforward computation shows that there exists some constant $c > 0$ s.t.

$$\begin{aligned} & -\partial_t \phi(t,x) + \lambda\phi(t,x) - \sup_{a\in A} b(x,a).\nabla\phi(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x,a)\nabla^2\phi(t,x)) \\ \geq \quad & e^{-\lambda t}\psi(x)(\lambda - c) \ . \end{aligned}$$

By taking $\lambda \geq c$, the function $\tilde{V}_\varepsilon := \tilde{V} + \varepsilon\phi$ is a supersolution to (9.1.1) for any $\varepsilon > 0$. Moreover, from the growth conditions on U, V and ϕ, we have

$$\lim_{|x|\to+\infty} \sup_{t\in[0,T]} (\tilde{U} - \tilde{V}_\varepsilon)(t,x) \quad = \quad -\infty \ . \tag{9.3.2}$$

for all $\varepsilon > 0$.

Step 3. We finally argue by contradiction to show that $\tilde{U} - \tilde{V}_\varepsilon \leq 0$ on $[0,T] \times \mathbb{R}^n$ for all $\varepsilon > 0$, which gives the required result by sending ε to 0. On the contrary, by continuity of $\tilde{U} - \tilde{V}_\varepsilon$, and from (9.3.2), there exists $\varepsilon > 0$, $(t,x) \in [0,T] \times \mathbb{R}^n$ such that

$$\sup_{[0,T]\times\mathbb{R}^n} (\tilde{U} - \tilde{V}_\varepsilon) \quad = \quad (\tilde{U} - \tilde{V}_\varepsilon)(t,x) \quad > \quad 0 \ . \tag{9.3.3}$$

Since $(\tilde{U} - \tilde{V}_\varepsilon)(T,.) \leq (\tilde{U} - \tilde{V})(T,.) \leq 0$ on $\mathbb{R}^n$, we have $t < T$. Therefore, the first and second order optimality conditions (9.3.3) give

$$\begin{aligned} \partial_t(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &\leq 0 \ , \\ \nabla(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &= 0 \ , \\ \nabla^2(\tilde{U} - \tilde{V}_\varepsilon)(t,x) &\leq 0 \ . \end{aligned}$$

By writing that $\tilde{U}$ (resp. $\tilde{V}_\varepsilon$) is a subsolution (resp. supersolution) to (9.3.1), and recalling that $\tilde{F}$ is nondecreasing in its last argument, we then deduce that

$$\begin{aligned}
& \lambda(\tilde{U} - \tilde{V}_\varepsilon)(t,x) \\
\leq\ & \tilde{F}(t,x,\partial_t \tilde{U}(t,x), \nabla \tilde{U}(t,x), \nabla^2 \tilde{U}(t,x)) \\
& -\tilde{F}(t,x,\partial_t \tilde{V}_\varepsilon(t,x), \nabla \tilde{V}_\varepsilon(t,x), \nabla^2 \tilde{V}_\varepsilon(t,x)) \\
\leq\ & 0
\end{aligned}$$

which contradicts (9.3.3).

Corollary 9.3.1 *The value function v is the unique viscosity solution to* (9.1.1) *in the HJB case satisfying* (9.1.1) *and having polynomial growth.*

Proof. We simply need to prove that v has polynomial growth. It actually follows from the polynomial growth assumption on f and g and Proposition (6.1.1).