

Chapter 7

Verification approach

7.1 The verification result

In this section, we give a criterion to check whether a function is the value function related to a given optimal control problem.

We define the following operator

$$\mathcal{L}^a \varphi(t, x) = \partial_t \varphi(t, x) + b(x, a) \cdot \nabla \varphi(t, x) + \frac{1}{2} \text{Tr}(\sigma(x, a) \sigma(x, a)^\top \nabla^2 \varphi(t, x))$$

for $a \in A$, $(t, x) \in [0, T) \times \mathbb{R}^n$ and $\varphi \in C^{1,2}([0, T) \times \mathbb{R}^n)$.

Theorem 7.1.1 *Suppose that there exists a $C^{1,2}([0, T] \times \mathbb{R}^n)$ function φ with polynomial growth such that*

$$\varphi(T, x) = g(x), \quad x \in \mathbb{R}^n.$$

Suppose moreover that the following statements hold.

(i) *There exists a measurable map $\hat{\alpha} : [0, T) \times \mathbb{R}^n \rightarrow A$ such that*

$$\sup_{a \in A} \mathcal{L}^a \varphi(t, x) + f(x, a) = \mathcal{L}^{\hat{\alpha}(t, x)} \varphi(t, x) + f(x, \hat{\alpha}(t, x)) = 0$$

for all $(t, x) \in [0, T) \times \mathbb{R}^n$.

(ii) *For any initial condition $(t, x) \in [0, T) \times \mathbb{R}^d$, the SDE (6.1.1) with $\alpha = \hat{\alpha}(\cdot, X_\cdot^{t, x, \alpha})$ admits a solution $X^{t, x, \hat{\alpha}}$.*

Then, $\varphi = v$ and $\hat{\alpha}(\cdot, X^{t,x,\hat{\alpha}})$ is an optimal control for the initial condition (t, x) .

Proof. From Itô's formula, (i) and (ii), we have

$$\varphi(t, x) = \mathbb{E} \left[g(X_T^{t,x,\hat{\alpha}}) + \int_t^T f(X_s^{t,x,\hat{\alpha}}, \hat{\alpha}(s, X_s^{t,x,\hat{\alpha}})) ds \right].$$

This gives $\varphi(t, x) \leq v(t, x)$. We now prove the reverse inequality. Let $\alpha \in \mathcal{A}_t$, τ_n the first time s such that $|X_s^{t,x,\alpha}| \geq n$ and $\theta_n = T \wedge \tau_n$ for $n \geq 1$. We have from Itô's formula

$$\begin{aligned} & \mathbb{E} \left[\varphi(\theta_n, X_{\theta_n}^{t,x,\alpha}) + \int_t^{\theta_n} f(X_s^{t,x,\alpha}, \alpha_s) ds \right] \\ &= \varphi(t, x) + \mathbb{E} \left[\int_t^{\theta_n} (\mathcal{L}^\alpha \varphi(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s)) ds \right]. \end{aligned}$$

From (i), we have

$$\mathcal{L}^\alpha \varphi(s, X_s^{t,x,\alpha}) + f(X_s^{t,x,\alpha}, \alpha_s) \leq 0, \quad s \in [t, T].$$

Therefore, we get from the polynomial growth of φ and Fatou's lemma

$$\begin{aligned} \varphi(t, x) &\geq \liminf_{n \rightarrow +\infty} \mathbb{E} \left[\varphi(\theta_n, X_{\theta_n}^{t,x,\alpha}) + \int_t^{\theta_n} f(X_s^{t,x,\alpha}, \alpha_s) ds \right] \\ &\geq \mathbb{E} \left[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha}, \alpha_s) ds \right]. \end{aligned}$$

Finally, $\hat{\alpha}$ is optimal as it realizes $\varphi(t, x)$.

7.2 Application: portfolio allocation problem in finite horizon

We consider an optimal investment in the framework of the Black-Scholes-Merton model over a finite horizon T . We suppose that the market is composed by two assets: a nonrisky asset S^0 and a risky one S . The nonrisky asset follows a deterministic interest rate $r > 0$ and has an initial value $S_0^0 > 0$. It therefore satisfy

$$S_t^0 = S_0^0 + \int_0^t r S_u du, \quad t \in [0, T].$$

The risky asset is defined by its deterministic initial condition $S_0 > 0$ and the dynamics

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

7.2. APPLICATION: PORTFOLIO ALLOCATION PROBLEM IN FINITE HORIZON 41

where μ and σ are constants with $\sigma > 0$ and W is a one dimensional Brownian motion.

An agent invests at any time t a proportion α_t of his wealth in a stock S and $1 - \alpha_t$ in S^0 . His wealth process X^α evolves according to

$$\begin{aligned} dX_t &= \frac{X_t \alpha_t}{S_t} dS_t + \frac{X_t (1 - \alpha_t)}{S_t^0} dS_t^0 \\ &= X_t (r + (\mu - r) \alpha_t) dt + X_t \alpha_t \sigma dW_t, \quad t \in [0, T]. \end{aligned}$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in A , that is supposed to be closed and convex, and such that $\mathbb{P}$ -a.s. $\int_0^T |\alpha_s|^2 ds < +\infty$. This integrability condition ensures the existence and uniqueness of a strong solution to the SDE governing the wealth process controlled by $\alpha \in \mathcal{A}$. Given a portfolio strategy $\alpha \in \mathcal{A}$, we denote by $X^{t,x,\alpha}$ the corresponding wealth process starting from an initial capital $X_t^{t,x,\alpha} = x > 0$ at time $t \in [0, T]$. We suppose that the preferences of the agent are described by a utility function U of CRRA type given by

$$U(x) = \frac{x^p}{p}, \quad x \geq 0$$

with $p \in (0, 1)$. The agent aims at maximizing the expected utility from terminal wealth at horizon T . The value function of the utility maximization problem is then defined by

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[U(X_T^{t,x,\alpha})],$$

for $(t, x) \in [0, T] \times \mathbb{R}_+$. The HJB equation is then given by

$$\sup_{a \in A} \{ \mathcal{L}^a v(t, x) + f(x, a) \} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n \quad (7.2.1)$$

with

$$\mathcal{L}^a v(t, x) = \partial_t v(t, x) + x(a\mu + (1 - a)r) \partial_x v(t, x) + \frac{1}{2} x^2 a^2 \sigma^2 \partial_{xx}^2 v(t, x)$$

for $(t, x) \in [0, T) \times \mathbb{R}^n$. The terminal condition is then given by

$$v(T, x) = U(x), \quad x \in \mathbb{R}_+ \quad (7.2.2)$$

We look for an explicit smooth solution φ to (7.2.1)-(7.2.2). We propose a candidate solution in the form

$$\varphi(t, x) = \phi(t)U(x), \quad (t, x) \in [0, T) \times \mathbb{R}^n,$$

for some positive function ϕ . By substituting into the HJB equation, we derive that ϕ should satisfy the ordinary differential equation

$$\begin{aligned} \phi'(t) + \rho\phi(t) &= 0, \quad t \in [0, T) \\ \phi(T) &= 1, \end{aligned}$$

where

$$\rho = p \sup_{a \in A} \left\{ a(\mu - r) + r - \frac{1}{2}a^2(1 - p)\sigma^2 \right\}.$$

We then obtain $\phi(t) = \exp(\rho(T - t))$. Hence, the function given by

$$\varphi(t, x) = \exp(\rho(T - t))U(x), \quad (t, x) \in [0, T) \times \mathbb{R}^n,$$

is a smooth solution to the HJB PDE. Furthermore, the function $a \in A \mapsto a(\mu - r) + r - \frac{1}{2}a^2(1 - p)\sigma^2$ is strictly concave on the closed convex set A , and thus attains its maximum at some constant $\hat{a}$. Moreover, the SDE associated to the constant control $\hat{a}$

$$dX_s = X_s(r + (\mu - r)\hat{a})ds + X_s\hat{a}\sigma dW_s,$$

admits a unique solution for a given initial condition (t, x) . From the verification theorem, we have $\varphi = v$. In the case where $A = \mathbb{R}$, $\hat{a}$ and ρ can be explicitly computed and we have

$$\hat{a} = \frac{\mu - r}{\sigma^2(1 - p)}$$

and

$$\rho = \frac{(\mu - r)^2}{2\sigma^2} \frac{p}{1 - p} + rp.$$

7.3 Application: investment-consumption problem

We use the framework of the previous section for the model on asset prices. A control is a pair of progressively measurable process (α, c) valued in $A \times \mathbb{R}_+$ for some closed convex subset A of $\mathbb{R}$ such that $\mathbb{P}$ -a.s. $\int_0^T |\alpha_t|^2 dt + \int_0^T c_t dt < +\infty$. We denote by $\mathcal{A} \times \mathcal{C}$ the set of control processes. The quantity α_t represents the proportion of wealth invested in stock S , and c_t is the consumption per unit of wealth. Given $(\alpha, c) \in \mathcal{A} \times \mathcal{C}$, there exists a unique solution, denoted by $X^{t,x}$, to the SDE governing the wealth process

$$dX_s = X_s(r + (\mu - r)\alpha_t - c_t)ds + X_s\alpha_t\sigma dW_s, \quad s \in [t, T],$$

given the initial condition $X_t = x \geq 0$. The agent's investment-consumption problem is to maximize over strategies (α, c) the expected utility from intertemporal consumption up to a random horizon T . Given a utility function u for consumption, we then consider the corresponding value function:

$$v(t, x) = \sup_{(\alpha, c) \in \mathcal{A} \times \mathcal{C}} \mathbb{E} \left[u(X_T^{t,x}) + \int_t^T u(c_s X_s^{t,x}) ds \right], \quad (t, x) \in [0, T] \times \mathbb{R}$$

The HJB equation associated to this control problem is

$$\sup_{(a, c) \in A \times \mathbb{R}_+} \{ \mathcal{L}^{(a, c)} v + u(cx) \} = 0$$

where $\mathcal{L}^{a, c} v(t, x) = \partial_t v(t, x) + x(a\mu + (1-a)r - c)\partial_x v(t, x) + \frac{1}{2}x^2 a^2 \sigma^2 \partial_{xx}^2 v(t, x)$ for $(t, x) \in [0, T] \times \mathbb{R}_+$. By defining $\tilde{u}(z) = \sup_{C \geq 0} [u(C) - Cz]$, $z \geq 0$, the Legendre transform of u , this HJB equation may be written as

$$\sup_{a \in A} \{ \mathcal{L}^a v \} + \tilde{u}(\partial_x v) = 0$$

with $\mathcal{L}^a$ defined as in the previous section. If we take $u(x) = x^p/p$ then $\tilde{u}(z) = z^{-q}/q$ with $q = p/(1-p)$. We next look for a candidate solution of the form

$$\varphi(t, x) = \phi(t)u(x)$$

By substituting into the HJB equation, we derive that ϕ should satisfy the ordinary differential equation

$$\begin{aligned} \phi'(t) + \rho\phi(t) + \frac{p}{q}\phi(t)^{-q} &= 0, \quad t \in [0, T] \\ \phi(T) &= 1, \end{aligned}$$

where

$$\rho = p \sup_{a \in A} \{a(\mu - r) + r - \frac{1}{2}a^2(1 - p)\sigma^2\}.$$

We notice that the ODE can be rewritten

$$\begin{aligned} (\phi^{q+1}(t))' + (q+1)\rho\phi^{q+1}(t) + \frac{(q+1)p}{q} &= 0, \quad t \in [0, T], \\ \phi^{q+1}(T) &= 1. \end{aligned}$$

Therefore we get

$$\phi(t)^{q+1} = \left(1 + \frac{p}{\rho q}\right)e^{(q+1)\rho(T-t)} - \frac{p}{\rho q}, \quad t \in [0, T].$$

We then have that the optimizer $\hat{a}$ in the variable a is the same as that of the previous section. We next compute the optimizer an the variable c and we have

$$\begin{aligned} \hat{c}^*(t, x) &= \frac{1}{x} (\phi(t)u'(x))^{\frac{1}{p-1}} \\ &= \left(\left(1 + \frac{p}{\rho q}\right)e^{\rho(q+1)(T-t)} - \frac{p}{\rho q} \right)^{-1} \end{aligned}$$

We observe that $c^*(t, x) = c^*(t) \in [0, 1]$. Therefore the related SDE

$$dX_s = X_s(r + (\mu - r)\hat{a} - c^*(t))ds + X_s\hat{a}\sigma dW_s, \quad s \in [t, T],$$

admits a unique solution. We can then apply the verification theorem to get that

$$v(t, x) = \left(\left(1 + \frac{p}{\rho q}\right)e^{\rho(q+1)(T-t)} - \frac{p}{\rho q} \right)^{\frac{1}{q+1}} \frac{x^p}{p}, \quad (t, x) \in [0, T] \times \mathbb{R}_+.$$