Chapter 6

Control of diffusion processes

6.1 The stochastic control problem

To define the control problem, we introduce the following elements.

Set of controls We fix a subset A of some $\mathbb{R}^p$ for $p \geq 1$. We suppose that A is bounded and we denote by $\mathcal{A}$ the set of $\mathbb{F}$-progressive processes $(\alpha_t)_{t\in[0,T}$ valued in $\mathcal{A}$. We also denote by $\mathcal{A}_t$ the subset of $\mathcal{A}$ whose elements are independent of $\mathcal{A}_t$.

Controlled diffusion process We fix two functions $b, \sigma :\ \mathbb{R}^n \times A \to \mathbb{R}^n, \mathbb{R}^{d\times n}$. We suppose that b and σ are continuous and there exists a constant L such that

$$|b(x,a) - b(x',a')| + |\sigma(x,a) - \sigma(x',a')| \quad \leq \quad L\big(|x-x'| + |a-a'|\big) \ ,$$

for all $(x,a),(x',a') \in \mathbb{R}^n \times A$. From Theorem 5.2.1, we have existence and uniqueness of the controlled process $X^{t,x,\alpha}$ defined by

$$X_s^{t,x,\alpha} \quad = \quad x + \int_t^s b(X_u^{t,x,\alpha},\alpha_u)du + \int_t^s \sigma(X_u^{t,x,\alpha},\alpha_u)dW_u \qquad (6.1.1)$$

for any $(t,x) \in [0,T] \times \mathbb{R}^n$ and any control process $\alpha \in \mathcal{A}$. We also have the following result.

Proposition 6.1.1 *For $p \geq 1$, there exists a constant C_p such that*

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[t,T]} |X_s^{t,x,\alpha}|^p\Big] &\leq C_p(1+|x|^p)\;,\\
\mathbb{E}\Big[\sup_{s\leq t\leq T} |X_s^{t,x,\alpha}-X_s^{t,x',\alpha}|^p\Big] &\leq C_p|x-x'|^p\;,\\
\mathbb{E}\Big[\sup_{t\leq s\leq T} |X_s^{t,x,\alpha}-X_{s\vee(t+h)}^{t+h,x,\alpha}|^p\Big] &\leq C_p h^{\frac{p}{2}}\big(1+|x|^p\big)\;,\\
\mathbb{E}\Big[\sup_{s\leq t\leq T} |X_s^{t,x,\alpha}-X_s^{t,x,\alpha'}|^p\Big] &\leq C_p\mathbb{E}\Big[\int_t^T |\alpha_s-\alpha'_s|^p ds\Big]\;,
\end{aligned}$$

for all $t \in [0,T]$, $h \in [0,T-t]$, $x, x' \in \mathbb{R}^n$ and $\alpha, \alpha' \in \mathcal{A}$.

Reward functions and gain. We fix two reward functions $f :\ \mathbb{R}^n \times A \to \mathbb{R}$ and $g :\ \mathbb{R}^n \to \mathbb{R}$. We assume that f and g are locally Lipschitz continuous, that is, for any $N > 0$, there exists a constant L_N such that

$$|f(x,a)-f(x',a')|+|g(x)-g(x')| \;\;\leq\;\; L_N\big(|x-x'|+|a-a'|\big)$$

for any $(x,a),(x',a') \in \mathbb{R}^n \times A$ such that $|x| \leq N$ and $|x'| \leq N$. We also assume that f and g have a polynomial growth, that is, there exist a constant C and an integer N such that

$$|f(x,a)|+|g(x)| \;\;\leq\;\; C\big(1+|x|^N\big) \tag{6.1.2}$$

for all $(x,a) \in \mathbb{R}^n \times A$. We next define the gain functional $J :\ [0,T]\times\mathbb{R}^n\times\mathcal{A} \to \mathbb{R}$ by

$$J(t,x,\alpha) \;\;=\;\; \mathbb{E}\Big[g(X_T^{t,x,\alpha})+\int_t^T f(X_s^{t,x,\alpha},\alpha_s)ds\Big]$$

for $(t,x,\alpha) \in [0,T]\times\mathbb{R}^n\times\mathcal{A}$. We observe that $J(t,x,\alpha)$ is well defined under the polynomial growth assumption from Proposition (6.1.1).

We now define the value function v of the considered stochastic control problem by

$$v(t,x) \;\;=\;\; \sup_{\alpha\in\mathcal{A}_t} J(t,x,\alpha) \tag{6.1.3}$$

for $(t,x) \in [0,T]\times\mathbb{R}^n$ where

$$\mathcal{A}_t \;\;=\;\; \{\alpha\in\mathcal{A}\;:\;\alpha \text{ independent of } \mathcal{F}_t\}\;,\quad t\in[0,T]\;.$$

Our goal is to characterize the function v in terms of a partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation.

6.2 Well posedness and regularity of the value function

Proposition 6.2.1 *The value function v is well defined in $\mathbb{R}$. More precisely, there exists a constant C such that*

$$|v(t,x)| \leq C(1+|x|^N)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

The proof follows from (6.1.2) and Proposition 6.1.1.

Proposition 6.2.2 *For $\Theta \subset [0,T] \times \mathbb{R}^n$ there exists a real map $\lambda_\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda_\Theta(r) \to 0$ as $r \to 0$ and*

$$|J(t,x,\alpha) - J(s,y,\alpha)| \leq \lambda_\Theta\big(|t-s| + |x-y|\big) \tag{6.2.1}$$

for all $(t,x),(s,y) \in \Theta$ and $\alpha \in \mathcal{A}$. The value function v is locally uniformly continuous and has polynomial growth.

Proof. Since f and g are locally Lipschitz continuous with polynomial growth, we get from Proposition (6.1.1) that there exists a constant C, an integer p and, for each N, there exists a constant C_N such that

$$\begin{aligned} & |J(t,x,\alpha) - J(s,y,\alpha)| \\ \leq\ & C_N\big(|t-s|^{\frac{1}{2}}(1+|x|) + |x-y| + |s-t|L_N \\ & + C(1+|x|^p+|y|^p)\mathbb{P}\big(\sup_{u\in[t,T]} \min\{|X_u^{t,x,\alpha}|, |X_{u\vee s}^{t,x,\alpha}|\} \geq N\big)\ , \end{aligned}$$

for any $t,s \in [0,T]$ $t \leq s$, any $x,y \in \mathbb{R}^n$ and any $\alpha \in \mathcal{A}$. Using Markov inequality, we get

$$\begin{aligned} & \sup_{\alpha\in\mathcal{A}} |J(t,x,\alpha) - J(s,y,\alpha)| \\ \leq\ & C_N\big(|t-s|^{\frac{1}{2}}(1+|x|) + |x-y| + |s-t|L_N \\ & + C(1+|x|^p+|y|^p)/N\ . \end{aligned}$$

which gives 6.2.1 and the local uniform continuity of v. The polynomial growth of v follows from the polynomial growth of f and g and Proposition 6.1.1.