Chapter 5

Stochastic differential equations

5.1 Definition

As we define the notion of ordinary differential equation in the deterministic case, we define here an object called **Stochastic Differential Equation** (SDE for short). To this end we consider two measurable functions μ and σ from $[0, T] \times \mathbb{R}$ into $\mathbb{R}$.

The SDE with **drift** coefficient μ and **diffusion** coefficient σ is then given by an initial condition and a dynamics as follows

$$X_0 \ = \ x \ , \qquad\qquad dX_t \ = \ \mu(t, X_t)dt \ + \ \sigma(t, X_t)dW_t \qquad t \in [0, T]. \quad (5.1.1)$$

We now give the precise definition of a solution to the SDE.

Definition 5.1.1 *A process X is a solution to the SDE* (5.1.1) *if it is an* $\mathcal{F}$***-adapted process*** *and satisfies*

$$\int_0^T |\mu(s, X_s)| ds \ + \ \int_0^T \sigma^2(s, X_s) ds \ < \ \infty \quad \mathbb{P} - a.s$$

and

$$X_t \ = \ x \ + \ \int_0^t \mu(s, X_s) ds \ + \ \int_0^t \sigma(s, X_s) dW_s \quad \mathbb{P} - a.s.$$

for all $t \in [0, T]$.

5.2 Itô Cauchy Lipschitz theorem

The previous equations do not necessary have solutions. To ensure the existence and uniqueness of solutions, we need two types of conditions.

The first one ensures the uniqueness through the Lipschitz property of the functions μ et σ:

Condition 1: There exists a constant $L > 0$ such that

$$|\mu(t,x) - \mu(t,y)| \; + \; |\sigma(t,x) - \sigma(t,y)| \quad \leq \quad L\,|x-y|$$

for all $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^2$.

The second condition ensures that the process does not explode on the time intervall $[0,T]$.

Condition 2: There exists a constant $M > 0$ such that

$$|\mu(t,0)| \; + \; |\sigma(t,0)| \quad \leq \quad M$$

for all $t \in [0,T]$.

Theorem 5.2.1 *Suppose that μ and σ satisfy **Condition 1** and **Condition 2**. Then SDE* (5.1.1) *admits a unique solution.*

This result is admitted. As for the deterministic case, the proof of this theorem is based on a fixed point argument.

Remark 5.2.1 In the particular case where μ and σ do not depend on the variable t, **Condition 2** is implied by **Condition 1** .

5.3 Link with PDEs

Let $X^{t,x}$ be the unique solution to the SDE

$$X_s = x + \int_t^s b_u(X_u)du + \int_t^s \sigma_u(X_u)dW_u, \quad u \in [t,T], \tag{5.3.1}$$

for $t \in [0,T]$ and $x \in \mathbb{R}^n$. We next define the operator $\mathcal{L}$ by

$$\mathcal{L}\varphi(t,x) \quad = \quad \lim_{h \to 0} \frac{\mathbb{E}[\varphi(X^{t,x}_{t+h})] - \varphi(x)}{h}$$

for any function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}\varphi$ is well defined. $\mathcal{L}$ is called the generator of the diffusion. From Ito's formula, $\mathcal{L}\varphi$ is well defined for any φ bounded C^2 with bounded derivatives and we have

$$\mathcal{L}\varphi(t,x) \quad = \quad b_t(x).\nabla\varphi(x) + \frac{1}{2}\text{Tr}(\sigma_t(x)\sigma_t(x)^\top\nabla^2\varphi(x))$$

for $t \in [0,T]$ and $x \in \mathbb{R}^n$. The generator provides a connection between diffusion processes and linear partial differential equations.

Proposition 5.3.1 *Suppose that the function v defined by*

$$v(t,x) \quad = \quad \mathbb{E}[g(X_T^{t,x})]\ , \quad (t,x) \in [0,T] \times \mathbb{R}^n\ ,$$

is in $C^{1,2}([0,T] \times \mathbb{R}^n)$. Then, v solves the partial differential equation:

$$\partial_t v + \mathcal{L}v \quad = \quad 0 \text{ on } [0,T) \times \mathbb{R}^n$$

with terminal condition

$$v(T,.) \quad = \quad g \text{ on } \mathbb{R}^n\ .$$

Proof. Fix $(t,x) \in [0,T] \times \mathbb{R}^n$ and let $\tau := T \wedge \inf\{s > t : |X_s^{t,x} - x| \geq 1\}$. By the tower property and the Markov property of $X^{t,x}$ we have

$$v(t,x) \quad = \quad \mathbb{E}\big[v(s \wedge \tau, X_{s\wedge\tau}^{t,x})\big]$$

for any $s \in [t,T]$. Since $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ we have from Ito's formula

$$\begin{aligned} 0 &= \mathbb{E}\Big[\int_t^{s\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] + \mathbb{E}\Big[\int_t^{s\wedge\tau} \nabla v.\sigma(u, X_u^{t,x})dW_u\Big] \\ &= \mathbb{E}\Big[\int_t^{s\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big] \end{aligned}$$

where the last inequality comes from the boundedness of $X^{t,x}$ on $[t,\tau]$. We now take $s = t + h$ with $h > 0$ and we get

$$0 \quad = \quad \mathbb{E}\Big[\frac{1}{h}\int_t^{(t+h)\wedge\tau} \big(\partial_t v + \mathcal{L}v\big)(u, X_u^{t,x})du\Big]\ .$$

From the mean value theorem and the dominated convergence theorem, we get the result by sending h to $0+$.

Feynman-Kac representation of Cauchy problem We consider the following linear partial differential equation called Cauchy problem

$$\begin{cases} \partial_t v + \mathcal{L}v - kv + f = 0, \text{ on } [0,T) \times \mathbb{R}^n, \\ v(T,.) = g\ , \text{ on } \mathbb{R}^n\ . \end{cases} \tag{5.3.2}$$

Here k and f are functions from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}$, The next results provides a representation of this purely deterministic problem by means of stochastic differential equations.

Theorem 5.3.1 *Assume that the coefficients b and σ satisfy **Condition 1** and **Condition 2**. Assume further that the function k is uniformly lower bounded, and f has quadratic growth in x uniformly in t. Let $v \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ be a solution of* (5.3.2) *with quadratic growth in x uniformly in t. Then*

$$v(t,x) \quad = \quad \mathbb{E}\left[\int_t^T \beta_s^{t,x} f(s, X_s^{t,x}) ds + \beta_T^{t,x} g(X_T^{t,x})\right]$$

where $X^{t,x}$ is the unique solution to (5.3.1) *and*

$$\beta_s^{t,x} \quad = \quad e^{-\int_t^s k(u, X_u^{t,x})}\;, \quad s \in [t,T]\;,$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$.

Proof. We introduce the sequence of stopping times $(\tau_n)_n$ defined by

$$\tau_n \quad = \quad \Big(T - \frac{1}{n}\Big) \wedge \inf\{s > t \;:\; |X_s^{t,x} - x| \;\geq\; n\}\;, \quad n \geq 1\;.$$

We notice that that $\tau_n \uparrow T-$ $\mathbb{P}$-a.s. as $n \uparrow +\infty$. We then have from Itô's formula

$$\begin{aligned} d\big(\beta_s^{t,x} v(s, X_s^{t,x})\big) \quad &= \quad \beta_s^{t,x}(\partial_t v + \mathcal{L}v - kv)(s, X_s^{t,x}) ds \\ &\quad + \beta_s^{t,x} \nabla v.\sigma(s, X_s^{t,x}) dW_s\;, \end{aligned}$$

for $s \in [t,T)$. Since v is a solution of (5.3.2), we get

$$d\big(\beta_s^{t,x} v(s, X_s^{t,x})\big) \quad = \quad -\beta_s^{t,x} f(s, X_s^{t,x}) ds + \beta_s^{t,x} \nabla v.\sigma(s, X_s^{t,x}) dW_s\;.$$

Therefore, we get

$$\begin{aligned} v(t,x) - \mathbb{E}\big[\beta_{\tau_n}^{t,x} v(\tau_n, X_{\tau_n}^{t,x})\big] \quad &= \quad \mathbb{E}\left[\int_t^{\tau_n} \beta_s^{t,x} f(s, X_s^{t,x}) ds\right] \\ &\quad - \mathbb{E}\left[\int_t^{\tau_n} \beta_s^{t,x} \nabla v.\sigma(s, X_s^{t,x}) dW_s\right]\;. \end{aligned}$$

We then notice that the expectation of the stochastic integral term vanishes as the stochastic integrand is bounded, the function v is regular and the function k is lower bounded. Therefore, we get

$$v(t,x) \quad = \quad \mathbb{E}\left[\beta_{\tau_n}^{t,x} v(\tau_n, X_{\tau_n}^{t,x}) + \int_t^{\tau_n} \beta_s^{t,x} f(s, X_s^{t,x}) ds\right]\;. \tag{5.3.3}$$

Using the continuity of v, the terminal condition of (5.3.2) and the continuity of $X^{t,x}$ we have

$$\beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds \xrightarrow[n\to+\infty]{\mathbb{P}-a.s.} \beta^{t,x}_T g(X^{t,x}_T) + \int_t^T \beta^{t,x}_s f(s, X^{t,x}_s) ds \ .$$

Moreover, since k is lower bounded the functions f and v have quadratic growth in x uniformly in t, there exists a constant C such that

$$\left| \beta^{t,x}_{\tau_n} v(\tau_n, X^{t,x}_{\tau_n}) + \int_t^{\tau_n} \beta^{t,x}_s f(s, X^{t,x}_s) ds \right| \ \leq \ C\Big(1 + \sup_{s\in[t,T]} |X^{t,x}_s|^2\Big) \ ,$$

for any $n \geq 1$. We can then apply the dominated convergence theorem and we get the result by sending n to ∞ in (5.3.3).