Chapter 4

Itô processes

4.1 Definition and properties

We introduce a new class of process with respect to which we can extend the stochastic integration.

Definition 4.1.1 *An **Ito process** is a process of the form*

$$X_t \;=\; X_0 \;+\; \int_0^t \varphi_s \, ds \;+\; \int_0^t \theta_s \, dB_s \;, \quad t \in [0,T] \;, \tag{4.1.1}$$

with X_0 $\mathcal{F}_0$-measurable, θ and φ two $\mathcal{F}$-adapted processes satisfying the integrability condition

$$\int_0^T |\theta_s|^2 ds < \infty \quad a.s. \qquad \textit{and} \qquad \int_0^T |\varphi_s| ds < \infty \quad a.s.$$

We also write

$$dX_t \;=\; \varphi_t \, dt \;+\; \theta_t \, dB_t$$

Proposition 4.1.1 *The decomposition* (4.1.1) *of an Ito process X with θ and φ satisfying condition (CI) is **unique**.*

Proposition 4.1.2 *The quadratic variation of an Ito process X with decomposition* (4.1.1) *is given by*

$$\langle X \rangle_t \;=\; \Big\langle \int_0^\cdot \theta_s \, dB_s \Big\rangle_t \;=\; \int_0^t \theta_s^2 \, ds$$

for all $t \in [0,T]$.

Proposition 4.1.3 *The quadratic covariation between two Ito processes X^1 et X^2 with decomposition*

$$dX_t^i \quad = \quad \varphi_t^i \, dt \; + \; \theta_t^i \, dB_t$$

is given by

$$\langle X^1, X^2 \rangle_t \quad = \quad \int_0^t \theta_s^1 \, \theta_s^2 \, ds$$

Definition 4.1.2 *Let X be an Ito process given by* (4.1.1) *with θ and φ satifying condition (CI). For process $\phi \in L^2_{\mathcal{F}}(\Omega, [0,T])$ satisfying*

$$\mathbb{E}\Big[\int_0^T |\phi_s \varphi_s| ds \Big] \quad < \quad \infty$$

and

$$\mathbb{E}\Big[\int_0^T |\phi_s \theta_s|^2 ds \Big] \quad < \quad \infty \; .$$

For $t \in [0,T]$, we define the stochastic integral of ϕ w.r.t. X on $[0,t]$ by

$$\boxed{\int_0^t \phi_s \, dX_s \;:=\; \int_0^t \phi_s \, \theta_s \, dB_s \;+\; \int_0^t \phi_s \, \varphi_s \, ds \; .}$$

4.2 Itô's formula

Theorem 4.2.1 *Let f be a C^2 function. Then we have*

$$\begin{aligned} f(X_t) \quad &= \quad f(X_0) \; + \; \int_0^t f'(X_s) \, dX_s \; + \; \frac{1}{2} \int_0^t f''(X_s) \, d\langle X \rangle_s \\ &= \quad f(X_0) \; + \; \int_0^t f'(X_s) \varphi_s \, ds \; + \; \int_0^t f'(X_s) \theta_s \, dB_s \; + \; \frac{1}{2} \int_0^t f''(X_s) \, \theta_s^2 ds \end{aligned}$$

for all $t \in [0,T]$. The infinitesimal notation of this relation is given by

$$df(X_t) \quad = \quad f'(X_t) \, dX_t \; + \; \frac{1}{2} f''(X_t) \, d\langle X \rangle_t$$

Proposition 4.2.1 (Stochastic integration by part formula) *Let X and Y be two Ito processes. Then we have*

$$X_t Y_t \quad = \quad X_0 Y_0 + \int_0^t X_s dY_s \; + \int_0^t Y_s dX_s \; + \int_0^t d\langle X, Y \rangle_s$$

for all $t \in [0,T]$. The infinitesimal notation for this formula is given by

$$d(XY)_t \quad = \quad X_t dY_t \; + \; Y_t dX_t \; + \; d\langle X, Y \rangle_t$$

Different forms for Ito's formula

1. For the Brownian motion with $f \in C^2(\mathbb{R})$:

$$f(W_t) \;=\; f(W_0) \;+\; \int_0^t f'(W_s)\,dW_s \;+\; \frac{1}{2}\int_0^t f''(W_s)\,ds$$

2. For an Ito process with $f \in C^2(\mathbb{R})$:

$$f(X_t) \;=\; f(X_0) \;+\; \int_0^t f'(X_s)\,dX_s \;+\; \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s$$

3. For an Ito process with $f \in C^{1,2}([0,T]\times\mathbb{R})$

$$f(t,X_t) \;=\; f(0,X_0) \;+\; \int_0^t \frac{\partial f}{\partial t}(s,X_s)\,ds \;+\; \int_0^t \frac{\partial f}{\partial x}(s,X_s)\,dX_s \;+\; \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)\,d\langle X\rangle_s$$

4. For a multi-dimensional Ito process with $f \in C^{1,2}([0,T]\times\mathbb{R}^d)$:

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)\,ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\,dX_s^i \\ &\quad + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)\,d\langle X^i, X^j\rangle_s \end{aligned}$$