Chapter 3

Itô's formula

3.1 The formula

We introduce the main skill to compute stochastic integrals without using approximation sequences of $\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$.

Theorem 3.1.1 *Let $f \in C^2(\mathbb{R})$ with bounded second order derivative. We then have*

$$f(B_t) \;=\; f(B_0) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds$$

for all $t \in [0,T]$. The infinitesimal notation of this formula is

$$d\,f(B_s) \;=\; f'(B_s)\,dB_s + \frac{1}{2} f''(B_s)\,ds\,.$$

Proof. Fix $t \in [0,T]$ and consider the subdivision $t_0 < t_1 < \cdots < t_n$ of the interval $[0,t]$ with $t_i = it/n$. From Taylor formula and the continuity of$B$ we have

$$\begin{aligned} f(B_t) - f(B_0) &= \sum_{i=1}^n [f(B_{t_i}) - f(B_{t_{i-1}})] \\ &= \sum_{i=1}^n f'(B_{t_i})(B_{t_i} - B_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^n f''(B_{\theta_i})(B_{t_i} - B_{t_{i-1}})^2 \end{aligned}$$

where θ_i is a random time valued in $]t_{i-1}, t_i[$ for each $i = 1, \ldots, n$. Since f' is differentiable with bounded derivative, we have from Proposition 2.2.3

$$\left\| \sum_{i=1}^n f'(B_{t_i})(B_{t_i} - B_{t_{i-1}}) - \int_0^t f'(B_s)\,dB_s \right\|_2 \xrightarrow[n\to\infty]{} 0$$

We now study the term

$$U_n \ := \ \sum_{i=1}^{n} f''(B_{\theta_i})(B_{t_i} - B_{t_{i-1}})^2$$

We aim at replacing θ_i by t_{i-1} in a first time and $(B_{t_i} - B_{t_{i-1}})^2$ by $t_i - t_{i-1}$ in a second time. Define the sequences $(V_n)_n$ and $(W_n)_n$ by

$$V_n \ := \ \sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 \qquad \text{et} \qquad W_n \ := \ \sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1})$$

From Cauchy-Schwartz inequality we have

$$\begin{aligned}
\mathbb{E}[|U_n - V_n|] \ &\leq \ \mathbb{E}\left[\sup_i |f''(B_{t_{i-1}}) - f''(B_{\theta_i})| \ \sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right] \\
&\leq \ \mathbb{E}\left[\sup_i |f''(B_{t_{i-1}}) - f''(B_{\theta_i})|^2\right]^{1/2} \ \mathbb{E}\left[\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right)^2\right]^{1/2} \\
& \qquad \mathbb{E}\left[\sup_{u,s\leq t \ , \ |u-s|\leq \frac{t}{n}} |f''(B_s) - f''(B_u)|^2\right]^{1/2} \ \mathbb{E}\left[\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right)^2\right]^{1/2}
\end{aligned}$$

The function $s \mapsto f''(B_s)$ is $\mathbb{P}$-a.s. continuous on the compact $[0,t]$, hence it is $\mathbb{P}$-a.s. uniformly continuous on $[0,t]$. Since f'' is bounded, we get from Lebesgue dominated convergence theorem that

$$\mathbb{E}\left[\sup_{u,s\leq t \ , \ |u-s|\leq \frac{t}{n}} |f''(B_s) - f''(B_u)|^2\right] \ \xrightarrow[n\to\infty]{} \ 0 \ .$$

For the second term we have from the definition of quadratic variation

$$\mathbb{E}\left[\left(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2\right)^2\right]^{1/2} \ \xrightarrow[n\to\infty]{} \ t \ .$$

We therefore deduce that a $\|U_n - V_n\|_1 \to 0$. We now study the difference $V_n - W_n$. We have

$$
\begin{aligned}
\mathbb{E}[|V_n - W_n|^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} f''(B_{t_{i-1}})((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}))\right)^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[\left(f''(B_{t_{i-1}})((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}))\right)^2\right] \\
&\leq \|f''\|_\infty^2 \sum_{i=1}^{n} Var((B_{t_i} - B_{t_{i-1}})^2) \\
&= \|f''\|_\infty^2 \sum_{i=1}^{n} 2(t_i - t_{i-1})^2 \;=\; 2\,\|f''\|_\infty^2 \frac{t^2}{n} \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

We deduce $\|V_n - W_n\|_2 \underset{n\to\infty}{\longrightarrow} 0$. Finally, from the definition of the Lebesgue integral and since f'' is bounded, we have

$$
\left\| W_n - \int_0^t f''(B_s)ds \right\|_1 \underset{n\to\infty}{\longrightarrow} 0
$$

The convergence in $L^2(\Omega)$ being stronger than the convergence in $L^1(\Omega)$, we finaly get

$$
\begin{aligned}
f(B_t) \;-\; f(B_0) &= \sum_{i=1}^{n} f'(B_{t_i})(B_{t_i} - B_{t_{i-1}}) \;+\; \frac{1}{2}\sum_{i=1}^{n} f''(B_{\theta_i})(B_{t_i} - B_{t_{i-1}})^2 \\
&\underset{n\to\infty}{\longrightarrow} \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds
\end{aligned}
$$

in $L^1(\Omega)$, which implies the $\mathbb{P}$-a.s. equality between these two random variables.

3.2 Extension of the SI

The assumption that f'' is bounded is restrictive and prevent us from applying Ito's formula to simplest financial models.

Actually, we can extend the definition of the stochastic integral w.r.t. B to the set $L^2_{\mathcal{F},loc}(\Omega,[0,T])$ including $L^2_{\mathcal{F}}(\Omega,[0,T])$ and defined by

$$
\begin{aligned}
L^2_{\mathcal{F},loc}(\Omega,[0,T]) \;:=\; \Big\{ &(\theta_s)_{0\leq s\leq T}\ \mathcal{B}([0,T])\otimes\mathcal{A}\text{-measurable process and} \\
&\mathcal{F}\text{-adapted s.t. } \int_0^T |\theta_s|^2 ds < \infty \quad \mathbb{P}-a.s. \Big\}
\end{aligned}
$$

By this generalisation, the martingale property of the stochastic integral is lost. It is replaced by a **local martingale** property. The Ito's formula can the be generalised as follows.

Theorem 3.2.1 *For any function $f \in C^2(\mathbb{R})$ we have*

$$f(B_t) \quad = \quad f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, ds \ , \quad \mathbb{P} - a.s.$$

for all $t \in [0, T]$.

We notice that in the previous formula, the stochastic integral appearing in the l.h.s. is well defined even if f'' is not bounded, thanks to the extension to $L^2_{\mathcal{F},loc}(\Omega, [0,T])$ of the stochastic integration.

3.3 Several forms for Itô's formula

1. For B one dimensional BM and $f \in C^2(\mathbb{R})$:

$$f(B_t) \quad = \quad f(B_0) \ + \ \int_0^t f'(B_s)\, dB_s \ + \ \frac{1}{2} \int_0^t f''(B_s)\, ds \ , \quad t \in [0,T] \ .$$

3. For B one dimensional BM and $f \in C^{1,2}([0,T] \times \mathbb{R})$ time dependent function:

$$\begin{aligned} f(t, B_t) \quad &= \quad f(0, B_0) \ + \ \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds \ + \ \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds \ , \quad t \in [0,T] \ . \end{aligned}$$

4. For $B = (B^1, \ldots, B^d)$ with $B^1, \ldots, B^d$ independent BM and $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$:

$$\begin{aligned} f(t, B_t) \quad &= \quad f(t, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, B_s)\, dB_s^i \\ &\quad + \frac{1}{2} \sum_{i=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i^2}(s, B_s)\, ds \end{aligned}$$