Chapter 2

The stochastic integral

We aim at defining an integral $\int_0^T \theta_s \, dB_s$ w.r.t. the Brownian motion B for a class of processes θ.

For a regular function g and a differentiable function f, the integral of g w.r.t. f is defined by

$$\int_0^T g(s)\, df(s) \quad = \quad \int_0^T g(s)\, f'(s) ds \; .$$

When f is not differentialble any more, this integral can be generalized to the case where f is a finite variation function as follow

$$\int_0^T g(s)\, df(s) \quad = \quad \lim_{\pi_n \to 0} \sum_{i=0}^{n-1} g(t_i)(f(t_{i+1}^n) - f(t_i^n)) \tag{2.0.1}$$

where $\Pi_n = \{t_0^n, \ldots, t_n^n\}$ is a sudivision of $[0, T]$ for all $n \geq 1$. This integral is called the Stieljes integral.

In our case, **we cannot use this definition of the integral since the Brownian motion is not a finite variation process**.

2.1 Stochastic integral on elementary processes

To give a sense to the integral w.r.t. the Brownian motion we use the fact that it has finite quadratic variation. This leads us to work in an L^2 type space. More precisely, we extend (2.0.1) by considering an L^2 sense for the limit:

$$\int_0^T \theta_s \, dB_s \quad = \quad \lim_{\pi_n \to 0} \sum_{i=0}^{n-1} \theta_{t_i}(B_{t_{i+1}} - B_{t_i})$$

Since we work with an L^2 sense, we ask the process θ to be square integrable and also adapted. We now give the precise definition of the square integrability we need. We fix in the sequel a complete right-continuous[1] filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We define the set $L^2_{\mathcal{F}}(\Omega, [0,T])$ by

$$L^2_{\mathcal{F}}(\Omega, [0,T]) \quad = \quad \Big\{(\theta_t)_{0\leq t\leq T} \quad \mathcal{F}\text{-adapted process cadlag s.t. } \theta \in L^2(\Omega, [0,T])\Big\}$$

We first construct the stochastic integral on a smaller set.

Définition 2.1.1 *A process $(\theta_t)_{0\leq t\leq T}$ is an* ***elementary process*** *if there exists a subdivision $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$ and a discrete time process $(\Theta_i)_{0\leq i\leq n-1}$ such that*

- *Θ_i is $\mathcal{F}_{t_i}$-measurable for all $i = 0, \ldots, n-1$,*
- *$\Theta_i \in L^2(\Omega)$ for all $i = 0, \ldots, n-1$,*
- *The following identification*

$$\theta_t(\omega) \quad = \quad \sum_{i=0}^{n-1} \Theta_i(\omega)\mathbb{1}_{]t_i,t_{i+1}]}(t) \tag{2.1.1}$$

  holds for all $t \in [0,T]$.

We denote by $\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$ the set of elementary processes. .

We notice that the set of elementary process is a subset of $L^2_{\mathcal{F}}(\Omega, [0,T])$:

$$\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T]) \quad \subset \quad L^2_{\mathcal{F}}(\Omega, [0,T]) \ .$$

Définition 2.1.2 *Let $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$ satisfying the decomposition* (2.1.1). *The stochastic integral between* 0 *and $t \in [0,T]$ of θ w.r.t. B is the random variable $\int_0^t \theta_s dB_s$ defined by*

$$\int_0^t \theta_s dB_s \quad := \quad \sum_{i=0}^{k} \Theta_i(B_{t_{i+1}} - B_{t_i}) + \Theta_k(B_t - B_{t_k})\,,$$

where k is such that $t \in]t_k, t_{k+1}]$. It can also be witten

$$\boxed{\int_0^t \theta_s dB_s \ = \ \sum_{i=0}^{n-1} \Theta_i(B_{t\wedge t_{i+1}} - B_{t\wedge t_i})\,.} \tag{2.1.2}$$

[1] *i.e.* $\cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for all $t \in [0,T)$.

To a process $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ we then associate the process $\int_0^{\cdot} \theta_s dB_s = \left(\int_0^t \theta_s dB_s\right)_{t\in[0,T]}$.

Remarque 2.1.1 According the Chasles relation we define $\int_s^t \theta_u$ by

$$\int_s^t \theta_u dB_u \;:=\; \int_0^t \theta_u dB_u - \int_0^s \theta_u dB_u$$

for any $s, t \in [0,T]$.

Proposition 2.1.1 *[Properties of the stochastic integral on $\mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$]*

(1) $\theta \mapsto \int_0^t \theta_s dB_s$ is a linear map on $\mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$.

(2) The process $\int_0^{\cdot} \theta_s dB_s$ is continuous for all $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$.

(3) The process $\int_0^{\cdot} \theta_s dB_s$ is $\mathcal{F}$-adapted for all $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$.

(4) $\mathbb{E}\left[\int_0^t \theta_s dB_s\right] = 0$ and $Var\left(\int_0^t \theta_s dB_s\right) = \mathbb{E}\left[\int_0^t \theta_s^2 ds\right]$ for all $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$.

(5) Isometry property:

$$\boxed{\mathbb{E}\left[\left(\int_0^t \theta_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t \theta_s^2 ds\right]}$$

for all $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ and all $t \in [0,T]$.

(6) More generaly, we have

$$\mathbb{E}\left[\int_s^t \theta_u dB_u|\mathcal{F}_s\right] = 0 \quad et \quad \mathbb{E}\left[\left(\int_s^t \theta_v dB_v\right)^2 |\mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \theta_v^2 dv|\mathcal{F}_s\right]$$

for all $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ and all $s, t \in [0,T]$ such that $s \leq t$.

(7) We also have the general result

$$\mathbb{E}\left[\left(\int_s^t \theta_v dB_v\right)\left(\int_s^u \phi_v dB_v\right)|\mathcal{F}_s\right] = \mathbb{E}\left[\int_s^{t\wedge u} \theta_v\phi_v dv|\mathcal{F}_s\right]$$

for all $\theta, \phi \in \mathcal{E}^2_{\mathbb{F}}(\Omega,[0,T])$ and all $t, s \in [0,T]$.

(8) The process $\left(\int_0^t \theta_s dB_s\right)_{0\leq t\leq T}$ is an $\mathcal{F}$-martingale for all $\theta \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$.

(9) The quadratic variation of the stochastic integral is given by

$$\boxed{\left\langle \int_0^\cdot \theta_s dB_s \right\rangle_t = \int_0^t \theta_s^2 ds}$$

for all $\theta \in \mathcal{E}^2_{\mathbb{F}}(\Omega, [0,T])$ *and all* $t \in [0,T]$.

(10) The quadratic covariation between two stochastic integrals given by

$$\boxed{\left\langle \int_0^\cdot \theta_s dB_s, \int_0^\cdot \phi_s dB_s \right\rangle_t = \int_0^t \theta_s \phi_s ds}$$

for all $\theta, \phi \in \mathcal{E}^2_{\mathbb{F}}(\Omega, [0,T])$ *and all* $t \in [0,T]$.

Example: We have

$$\int_0^t dB_s \quad = \quad B_t - B_0 = B_t$$

for all $t \in [0,T]$.

2.2 Extension of the SI

We notice that the stochastic integral of a process in $\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$ is a square integrable $\mathcal{F}$-martingale. We therefore define the set $\mathcal{M}^2([0,T])$ by

$$\mathcal{M}^2([0,T]) \quad := \quad \left\{ M \ \mathcal{F}\text{-martingale such that } \mathbb{E}[M_t^2] < \infty \quad \text{for all } t \in [0,T] \right\}$$

As defined now, the stochastic integration is a linear map from $\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$ to $\mathcal{M}^2([0,T])$. We now extend the stochastic integration to $L^2_{\mathcal{F}}(\Omega, [0,T])$. To this end we need the following lemma.

Lemme 2.2.1 *The set of elementary processes* $\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$ *is dense in* $L^2_{\mathcal{F}}(\Omega, [0,T])$ *for the norm* $\|\cdot\|_2'$. *In other words, for any* $\theta \in L^2_{\mathcal{F}}(\Omega, [0,T])$, *there exists a sequence* $(\theta^n)_n$ *in* $\mathcal{E}^2_{\mathcal{F}}(\Omega, [0,T])$ *such that*

$$\|\theta^n - \theta\|_2' \quad = \quad \mathbb{E}\left[\int_0^T (\theta_s - \theta_s^n)^2 \, ds\right]^{1/2} \quad \longrightarrow \quad 0 \qquad \textit{as} \quad n \to \infty \, .$$

The extension of the stochastic integral is given by the following result.

Theorem 2.2.1 *There exists a unique linear map I from $L^2_{\mathcal{F}}(\Omega,[0,T])$ to $\mathcal{M}^2([0,T])$ which coincides with the stochastic integral on $\mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ and satisfies the isometry property:*

$$\mathbb{E}\big[I(\theta)_t^2\big] \;=\; \mathbb{E}\left[\int_0^t \theta_s^2\,ds\right]$$

for all $\theta \in L^2_{\mathcal{F}}(\Omega,[0,T])$ and all $t \in [0,T]$.

Proof.

Approximation: Fix $\theta \in \mathcal{L}^2_{\mathcal{F}}(\Omega,[0,T])$. From the previous Lemma, There exists a sequence $(\theta^n)_n$ in $\mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ such that

$$\|\theta-\theta^n\|_2' \;=\; \mathbb{E}\left[\int_0^T(\theta_s-\theta_s^n)^2\,ds\right]^{1/2} \;\longrightarrow\; 0\,.$$

Convergence: From the isometry property between 0 and $t \in [0,T]$ applied to the process $(\theta^{n+p}\,-\,\theta^n) \in \mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ we have

$$\mathbb{E}\left[\left(\int_0^t(\theta_s^{n+p}\,-\,\theta_s^n)dB_s\right)^2\right] \;=\; \mathbb{E}\left[\int_0^t(\theta_s^{n+p}\,-\,\theta_s^n)^2ds\right] \;\leq\; \mathbb{E}\left[\int_0^T(\theta_s^{n+p}\,-\,\theta_s^n)^2ds\right]\,,$$

which can be rewrtitten

$$\left\|\int_0^t\theta_s^{n+p}dB_s\,-\,\int_0^t\theta_s^n dB_s\right\|_2 \;\leq\; \left\|\theta^{n+p}\,-\,\theta^n\right\|_2'\,. \tag{2.2.1}$$

Since $(\theta^n)_n$ converges in $L^2(\Omega,[0,T])$, it is a Cauchy sequence. From (2.2.1), the sequence $(\int_0^t\theta_s^n dB_s)_n$ is Cauchy in $L^2(\Omega)$. $L^2(\Omega)$ endowed with $\|.\|_2$ being a Banach space, the sequence $(\int_0^t\theta_s^n dB_s)$ converges in $L^2(\Omega)$. If we denote by $\int_0^t\theta_s dB_s$ its limit, we get by sending p to infinity in (2.2.1)

$$\mathbb{E}\left[\left(\int_0^t\theta_s\,dB_s\,-\,\int_0^t\theta_s^n dB_s\right)^2\right] \;=\; \mathbb{E}\left[\int_0^t(\theta_s\,-\,\theta_s^n)^2ds\right]\,.$$

Uniqueness: Suppose that $(\theta_n)_n$ et $(\phi_n)_n$ are two approximating sequences in $\mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ for $\theta \in L^2_{\mathcal{F}}(\Omega,[0,T])$. From the isometry property, we have

$$\left\|\int_0^t\theta_s^n\,dB_s\,-\,\int_0^t\phi_s^n dB_s\right\|_2 \;=\; \mathbb{E}\left[\int_0^t(\theta_s^n\,-\,\phi_s^n)^2ds\right]^{1/2} \;\leq\; \|\theta^n\,-\,\phi^n\|_2'$$

Therefore, by sending n to infinity we get that $\int_0^t\theta_s^n\,dB_s\,-\,\int_0^t\phi_s^n dB_s$ in $L^2(\Omega)$.

We now extend the properties of the stochastic integral to $L^2_{\mathcal{F}}(\Omega,[0,T])$.

Proposition 2.2.1 (Properties of the extended stochastic integral) *The stochastic integral satisfies the following properties on $L^2_{\mathcal{F}}(\Omega, [0,T])$:*

(1) $\theta \mapsto \int_0^t \theta_s dB_s$ *is a linear map on* $L^2_{\mathcal{F}}(\Omega, [0,T])$.

(2) The process $\int_0^\cdot \theta_s dB_s$ *is continuous for all* $\theta \in L^2_{\mathcal{F}}(\Omega, [0,T])$.

(3) The process $\int_0^\cdot \theta_s dB_s$ *is* $\mathcal{F}$*-adapted for all* $\theta \in L^2_{\mathcal{F}}(\Omega, [0,T])$.

(4) $\mathbb{E}\left[\int_0^t \theta_s dB_s\right] = 0$ *and* $Var\left(\int_0^t \theta_s dB_s\right) = \mathbb{E}\left[\int_0^t \theta_s^2 ds\right]$ *for all* $\theta \in L^2_{\mathcal{F}}(\Omega, [0,T])$.

(5) Isometry property:

$$\boxed{\mathbb{E}\left[\left(\int_0^t \theta_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t \theta_s^2 ds\right]}$$

for all $\theta \in \mathbb{L}^2_{\mathcal{F}}(\Omega, [0,T])$ *and all* $t \in [0,T]$.

(6) More generaly, we have

$$\mathbb{E}\left[\int_s^t \theta_u dB_u | \mathcal{F}_s\right] = 0 \quad et \quad \mathbb{E}\left[\left(\int_s^t \theta_v dB_v\right)^2 | \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \theta_v^2 dv | \mathcal{F}_s\right]$$

for all $\theta \in \mathbb{L}^2_{\mathcal{F}}(\Omega, [0,T])$ *and all* $s, t \in [0,T]$ *such that* $s \leq t$.

(7) We also have the general result

$$\mathbb{E}\left[\left(\int_s^t \theta_v dB_v\right)\left(\int_s^u \phi_v dB_v\right) | \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^{t\wedge u} \theta_v \phi_v dv | \mathcal{F}_s\right]$$

for all $\theta, \phi \in \mathcal{L}^2_{\mathbb{F}}(\Omega, [0,T])$ *and all* $t, s \in [0,T]$.

(8) The process $\left(\int_0^t \theta_s dB_s\right)_{0\leq t\leq T}$ *is an* $\mathcal{F}$*-martingale for all* $\theta \in \mathbb{L}^2_{\mathcal{F}}(\Omega, [0,T])$.

(9) The quadratic variation of the stochastic integral is given by

$$\boxed{\left\langle \int_0^t \theta_s dB_s \right\rangle = \int_0^t \theta_s^2 ds}$$

for all $\theta \in \mathcal{L}^2_{\mathbb{F}}(\Omega, [0,T])$ *and all* $t \in [0,T]$.

(10) The quadratic covariation between two stochastic integrals given by

$$\boxed{\left\langle \int_0^{\cdot} \theta_s dB_s, \int_0^{\cdot} \phi_s dB_s \right\rangle_t = \int_0^t \theta_s \phi_s ds}$$

for all $\theta, \phi \in L^2_{\mathbb{F}}(\Omega, [0,T])$ and all $t \in [0,T]$.

The case of deterministic process θ.

Proposition 2.2.2 *Suppose that the process θ is a deterministic function f of the time variable. Then the stochastic integral $\int_0^t f(s)dB_s$ is au Gaussian random variable called the **Wiener integral** with law given by*

$$\int_0^t f(s)dB_s \sim \mathcal{N}\left(0, \int_0^t f^2(s)ds\right) .$$

Proof. Indeed, $\int_0^t f(s)dB_s$ can be written as the limit in $L^2(\Omega)$ of random variables of the form

$$\sum_{i=0}^{n-1} f(t_i^n)(B_{t_{i+1}^n} - B_{t_i^n})$$

where $t_i^n = \frac{i}{n}t$ for $i = 1, \ldots, n$ and $n \geq 1$. We notice that $\sum_{i=0}^{n-1} f(t_i^n)(B_{t_{i+1}^n} - B_{t_i^n})$ As a limit in $L^2(\Omega)$ of Gaussian random variable, $\int_0^t f(s)dB_s$ is also a Gaussian random variable Since the convergence in $L^2(\Omega)$ implies the convergence and the expectation and the variance we get

$$\mathbb{E}\Big[\int_0^t f(s)dB_s\Big] = 0 \quad \text{and} \quad Var\Big[\int_0^t f(s)dB_s\Big] = \int_0^t f^2(s)ds .$$

Moreover, any linear combination of the $\int_0^{t_i} f(s)dB_s$ is also a Wiener integral and hence a Gaussian random variable. Therefore the process $\int_0^{\cdot} f(s)dB_s$ is a **Gaussian process** with covariance function given by

$$Cov\left(\int_0^t f(s)dB_s, \int_0^u g(s)dB_s\right) = \int_0^{t\wedge u} f(s)g(s)ds .$$

We can also prove that the random variable $\int_s^t f(u)dB_u$ is independent of $\sigma(B_u : u \leq s)$.

The case where the process θ is a deterministic function $f(B)$ of the Brownian motion.

We have seen that for a sequence (θ^n) in $\mathcal{E}^2_{\mathcal{F}}(\Omega,[0,T])$ converging to θ in $L^2(\Omega,[0,T])$, the stochastique integral $\int \theta dB$ is the limit in $L^2(\Omega)$ of the sequence $(\int \theta^n dB)_n$. In the case where theta is of the form $f(B)$. A natural sequence to approach the stochastic integral $\int_0^T f(B)dB$ is the following

$$\sum_{i=0}^{n-1} f(B_{t_i^n})(B_{t_{i+1}^n} - B_{t_i^n})$$

where $t_i^n = \frac{i}{n}T$ for $i = 1, \dots, n$ and $n \geq 1$. We study the conditions under which we have convergence.

Proposition 2.2.3 *Suppose that the function f admits a bounded first order derivative. Then we have then following convergence in $L^2(\Omega)$:*

$$\sum_{i=0}^{n-1} f(B_{\frac{i}{n}T})(B_{\frac{(i+1)}{n}T} - B_{\frac{i}{n}T}) \xrightarrow{L^2} \int_0^T f(B_s)dB_s \qquad as \quad n \to \infty.$$

Example: how to compute $\int_0^T B_s\, dB_s$?

Using the process $B^n_. := \sum_{i=0}^{n-1} B_{t_i}\mathbb{1}_{]t_i,t_{i+1}]}(.)$, The previous Proposition gives the convergence in $L^2(\Omega)$ of $\int_0^T B^n_s\, dB_s$ to $\int_0^T B_s\, dB_s$. We then notice that

$$\begin{aligned} 2\int_0^T B^n_s\, dB_s &= 2\sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}} - B_{t_i})\, ds = \sum_{i=0}^{n-1}(B^2_{t_{i+1}} - B^2_{t_i}) - \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2 \\ &= B^2_T - \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2 \end{aligned}$$

The second term in the r.h.s converges in $L^2(\Omega)$ to the quadratic variation of B on $[0,T]$. We finally get

$$B^2_T = \int_0^T 2B_s\, dB_s + T .$$

In particular, $(B^2_t - t)_{t\geq 0}$ is a martingale.

Loosely speaking, one can say that a small variation dB_t of B corresponds to a variation $\sqrt{dt}$ since its quadratic variation is dt $i.e.$

$$dB_t \sim \sqrt{dt} \Rightarrow (dB_t)^2 \sim dt$$

2.3 Girsanov transformation

Absolutely continuous change of probability

Consider a probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$. If $\widehat{\mathbb{P}}$ is absolutely continuous w.r.t. $\mathbb{P}$, The Radon Nikodym Theorem ensures the existence of an $\mathcal{F}_T$-measurable random variable Z_T such that $d\widehat{\mathbb{P}} = Z_T \, d\mathbb{P}$, i.e.

$$\widehat{\mathbb{P}}(A) \;=\; \int_A Z_T \qquad \forall A \in \mathcal{F}_T \;.$$

Saying that $\widehat{\mathbb{P}}$ and $\mathbb{P}$ are equivalent remains to say that they have the same support and hence Z_T do not vanish, i.e. $Z_T > 0$. Then, the Radon Nikodym density of $\mathbb{P}$ w.r.t. $\widehat{\mathbb{P}}$ is $1/Z_T$.

Since $\widehat{\mathbb{P}}$ is probability measure we have

$$\widehat{\mathbb{P}}(\Omega) \;=\; \mathbb{E}^{\mathbb{P}}[Z_T] \;=\; 1$$

From Bayes formula we know that for any $\mathcal{F}_T$-measurable random variable. X_T we have

$$\mathbb{E}^{\widehat{\mathbb{P}}}[X_T] \;=\; \mathbb{E}^{\mathbb{P}}[Z_T \, X_T]$$

To the random variable Z_T we associate the martingale process $(Z_t)_{0 \leq t \leq T}$ defined by

$$Z_t \;:=\; \mathbb{E}^{\mathbb{P}}[Z_T | \mathcal{F}_t]$$

for all $t \in [0, T]$. We then have for any $t \in [0, T]$ and any $\mathcal{F}_t$-measurable random variable X_t

$$\mathbb{E}^{\widehat{\mathbb{P}}}[X_t] \;=\; \mathbb{E}^{\mathbb{P}}[Z_T \, X_t] \;=\; \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Z_T \, X_t | \mathcal{F}_t]] \;=\; \mathbb{E}^{\mathbb{P}}[Z_t \, X_t] \;.$$

Actually Z_t is the Radon Nikodym density on $\mathcal{F}_t$ of $\widehat{\mathbb{P}}$ w.r.t. $\mathbb{P}$. We denote by

$$Z_t \;=\; \left.\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$$

For an $\mathcal{F}$-adapted process X, the generalized Bayes formula is given by

$$\mathbb{E}^{\widehat{\mathbb{P}}}[X_T | \mathcal{F}_t] \;=\; \mathbb{E}^{\mathbb{P}}\left[\frac{Z_T}{Z_t} X_T | \mathcal{F}_t\right] \;=\; \frac{1}{Z_t} \mathbb{E}^{\mathbb{P}}\left[Z_T \, X_T | \mathcal{F}_t\right] \;.$$

Indeed, for any bounded $\mathcal{F}_t$-measurable random variable Y_t we have

$$\mathbb{E}^{\widehat{\mathbb{P}}}[X_T \, Y_t] \;=\; \mathbb{E}^{\mathbb{P}}[Z_T \, X_T \, Y_t] \;=\; \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Z_T \, X_T | \mathcal{F}_t] \, Y_t] \;=\; \mathbb{E}^{\widehat{\mathbb{P}}}\left[\frac{1}{Z_t} \mathbb{E}^{\mathbb{P}}[Z_T \, X_T | \mathcal{F}_t] \, Y_t\right] \;.$$

Theorem 2.3.1 (Girsanov Theorem) *There exists a probability measure $\widehat{\mathbb{P}}$ equivalent to the historical probability $\mathbb{P}$ defined on $(\Omega, \mathcal{F}_T)$ by*

$$\left.\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} \;:=\; Z_T \;:=\; e^{-\lambda W_T - \frac{\lambda^2}{2}T}$$

under which the process $\widehat{W}$ defined by

$$\widehat{W}_t \;:=\; W_t + \lambda t$$

for all $t \in [0, T]$, is a Brownian motion.

Proof. $\widehat{W}$ is a continuous process and $\widehat{W}_0 = 0$. Thus it is a Brownian motion if for any $n \geq 1$, any $0 \leq t_1 \leq \ldots \leq t_n = T$, the random variables $\widehat{W}_{t_i} - \widehat{W}_{t_{i-1}}$ are independent and with respective laws $\mathcal{N}(0, t_i - t_{i-1})$. To prove this it sufficies to show that for any $u_1, \ldots, u_n$ de $\mathbb{R}^n$, we have the following identification between the characteristic functions

$$\mathbb{E}^{\widehat{\mathbb{P}}}\left[e^{i\sum_{j=1}^n u_j(\widehat{W}_{t_j} - \widehat{W}_{t_{j-1}})}\right] \;=\; \prod_{j=1}^n e^{-\frac{u_i^2}{2}(t_j - t_{j-1})}$$

This identification is proved as follows

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{P}}}\left[e^{i\sum_{j=1}^n u_j(\widehat{W}_{t_j} - \widehat{W}_{t_{j-1}})}\right] &= \mathbb{E}^{\mathbb{P}}\left[\prod_{j=1}^n e^{iu_j[(W_{t_j} - W_{t_{j-1}}) + \lambda(t_j - t_{j-1}))]} e^{-\lambda W_T - \frac{\lambda^2}{2}T}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\prod_{j=1}^n e^{(iu_j - \lambda)(W_{t_j} - W_{t_{j-1}}) + (i\lambda u_j - \frac{\lambda^2}{2})(t_j - t_{j-1})}\right] \\
&= \prod_{j=1}^n e^{(i\lambda u_j - \frac{\lambda^2}{2})(t_j - t_{j-1})} \mathbb{E}^{\mathbb{P}}\left[e^{(iu_j - \lambda)(W_{t_j} - W_{t_{j-1}})}\right] \\
&= \prod_{j=1}^n e^{(i\lambda u_j - \frac{\lambda^2}{2})(t_j - t_{j-1})} e^{\frac{(iu_j - \lambda)^2}{2}(t_j - t_{j-1})} \\
&= \prod_{j=1}^n e^{-\frac{u_j^2}{2}(t_j - t_{j-1})} .
\end{aligned}$$

□