Chapter 1

The Brownian motion

We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

1.1 Continuous time processes, martingale and filtration

Définition 1.1.1 *A **(random) process** X is a family of random variables $(X_t)_{t\in[0,T]}$.*

Définition 1.1.2 *A **Filtration** is a nondecreasing family $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-aglebra of $\mathcal{A}$, that is*

$$\mathcal{F}_s \;\subset\; \mathcal{F}_t \subset \mathcal{A}$$

for all $s, t \in [0, T]$ such that $s \leq t$.

$\mathcal{F}_t$ represents the information available at time t. It is therefore natural to suppose it to be nondecreasing with time. For a process X we define the filtration generated by X $\mathcal{F}^X$ by

$$\mathcal{F}^X_t \;=\; \sigma(X_u,\ u \leq t)\ , \quad t \geq 0\ .$$

Définition 1.1.3 *A process $(X_t)_{t\in[0,T]}$ is said to be $\mathcal{F}$**-adapted** if X_t is $\mathcal{F}_t$-mesurable for all $t \in [0, T]$.*

Définition 1.1.4 *A propcess $M = (M_t)_{t\in[0,T]}$ is an $\mathcal{F}$**-martingale** if* *(i)* M *est* $\mathcal{F}$*-adapted,*

(ii) $\mathbb{E}[|M_t|] < \infty$*, for all* $t \in [0, T]$*,*

(iii) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ *for all* $s, t \in [0, T]$ *such that* $s \leq t$*.*

A process M *is a* $\mathcal{F}$***-super-martingale*** *(resp.* $\mathcal{F}$***-sub-martingale****) if it satisfies properties (i) et (ii) and* $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$ *(resp.* $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$*) for all* $s, t \in [0, T]$ *such that* $s \leq t$*.*

When $\mathcal{F}$ is the filtration generated by M, we simply say that M is a martingale.

1.2 Gaussian processes and Brownian motion

Définition 1.2.1 *A random vector* $(X_1, \ldots, X_n)$ *is a* ***gaussian vector*** *if any linear combination of the components* X_i *is gaussian, that is, the random variable* $\langle a, X\rangle$ *defined by*

$$\langle a, X\rangle \quad := \quad \sum_{k=1}^{n} a_i X_i$$

is. gaussian for all $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$*.*

We provide two useful propositions to manipulate manipulation gaussian vectors.

Proposition 1.2.1 *If* (X_1, X_2) *is gaussian,* X_1 *et* X_2 *are independent if and only if* $cov(X_1, X_2) = 0$*.*

Proposition 1.2.2 *Any random vector with gaussian and indépendant components is a gaussian vector.*

We turn to the continuous version of gaussian vectors that are gaussian processes.

Définition 1.2.2 *A process* $(X_t)_{t\in[0,T]}$ *is a* ***gaussian process*** *if for an y* n *any* $t_1, \ldots, t_n \in [0, T]$*,* $(X_{t_1}, \ldots, X_{t_n})$ *is a gaussian vector.*

Définition 1.2.3 *A* ***(standard) Brownian motion*** *is a process* B *satisfying:*

(i) $B_0 = 0$ $\mathbb{P}$*-a.s.,*

(ii) B *is continuous, i.e.* $t \mapsto B_t(\omega)$ *is continuous for* $\mathbb{P}$*-almost all* $\omega \in \Omega$*,*

(iii) B *has independent increments:* $B_t - B_s$ *is independent of* $\sigma(B_u,\ u \leq s)$ *for* $t, s \in [0, T]$ *such that* $s \leq t$*,*

(iv) B *has stationary and gaussian increments:* $B_t - B_s \sim \mathcal{N}(0, t-s)$ *for* $t, s \in [0, T]$ *such that* $s \leq t$*.*

Théorème 1.2.1 *There exists a probability space endowed with a process* B *such that* B *is a Brownian motion*

This result is admitted. The main issue is to prove cotninuity ofof the candidate. The Brownian motion can also be seen as the scaling limit of a random walk.

Proposition 1.2.3 *If* B *is a Brownian Motion the processes* (B_t)*,* $(B_t^2 - t)$ *and* $(e^{\sigma B_t - \frac{\sigma^2 t}{2}})$ *(Brownien Exponentiel) are martingales w.r.t. the filtration of* B*.*

Proof.

$\square$

Proposition 1.2.4 *If B is a Brownian motion, the processes processus $\frac{1}{a}B_{a^2t}$, $tB_{1/t}$ + value 0 at $t = 0$, and $B_{t+t_0} - B_t$ are Brownian motions.*

Proof.

□

We now provide some qualitative properties to better understand the behavior of the the Brownian motion.
(1) If B is a Brownian motion then :

$$\frac{B_t}{t} \underset{t\to\infty}{\longrightarrow} 0 \quad p.s.$$

This result is easy to obtain for t valued in $\mathbb{N}$:

$$\frac{B_n}{n} = \frac{1}{n}\sum_{i=1}^{n}(B_i - B_{i-1}) \underset{n\to\infty}{\longrightarrow} \mathbb{E}[B_i - B_{i-1}] = 0 \quad p.s.$$

grâce à la loi de grands nombres, les $B_i - B_{i-1}$ étant indépendantes de loi $\mathcal{N}(0,1)$.
(2) The trajectories of the Brownian motion are continuous but not differentiable at any point $t \in \mathbb{R}_+$.
(3) If B is a Brownian motion, then we have

$$\overline{\lim_{t\to\infty}}\, B_t = +\infty \qquad \text{et} \qquad \underline{\lim_{t\to\infty}}\, B_t = -\infty$$

(4) Le mouvement brownien mettes any value at an infinity of times.
(5) The Brownian motion is Markov

$$\forall s \leq t \qquad B_t|\mathcal{F}_s \overset{loi}{\sim} B_t|B_s$$

1.3 Variation of the Brownian motion

Définition 1.3.1 (Variation over a grid) *Fix $t \geq 0$. For a grid $\pi = \{t_0 = 0 < t_1 < \cdots < t_n = t\}$, the variation $V_t^\pi(X)$ of a process X over the grid π is defined by*

$$V_t^\pi(X) = \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}| .$$

Proposition 1.3.1 *For a time $t \in \mathbb{R}_+$ and a process X, $V_t^\pi(X)$ is non-decreasing in π. Therefore the limit*

$$V_t(X) = \lim_{|\pi|\to 0} V_t^\pi(X)$$

exists and is called the total variation of X.

$V_t(X)$ is actually the length of the path between 0 and t of a particle whose position over time is given by the process X.

Proposition 1.3.2 *If B is a Brownian motion, we have $V_t(B) = +\infty$ $\mathbb{P}$-a.s. for all $t > 0$.*

As the total length of a Brownian motion is infinite, we may look for a better suited quantity to describe the length of its trajectory.

Définition 1.3.2 (Quadratic variation) *Fix $t \geq 0$. For a grid $\pi = \{t_0 = 0 < t_1 < \cdots < t_n = t\}$, the quadratic variation $V_t^{2,\pi}(X)$ of a process X over the grid π is defined by*

$$V_t^{2,\pi}(X) = \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|^2 .$$

If the limit (in some sense)

$$\langle X \rangle_t = \lim_{|\pi|\to 0} V_t^\pi(X)$$

exists in $\mathbb{R}_+$, it is called the quadratic variation of X.

Proposition 1.3.3 *(i) If X is a process such that $V_t(X) < +\infty$, then $\langle X \rangle_t = 0$.*

(ii) If B is a Brownian motion, we have $\langle B \rangle_t = t$ for $t \in \mathbb{R}_+$.