Stochastic calculus and control - tutorial

Stochastic Control

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We start by extending the definition of the Brownian motion to the vector case.

Definition. Let $(W_t)_{t\geq 0}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{F}$ its canonical filtration. W is an $\mathbb{F}-$Brownian motion if

(i) $W_0 = 0$,
(ii) For all $0 \leq s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$,
(iii) For all $0 \leq s \leq t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$.

Definition. Let $(W_t)_{t\geq 0}$ be an $\mathbb{R}^d-$valued stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. W is a Brownian motion if the components $W^i, i \in \{1 \dots d\}$ are independent Brownian motions, i.e. (i), (ii) hold and for all $0 \leq s \leq t$, $W_t - W_s \sim \mathcal{N}(0, (t-s)I_d)$ where I_d is the indentity matrix of dimension d.

1. Portfolio allocation with multiple assets

Consider a market with d risky assets $S = (S^1, \dots, S^d)$ with (multi-dimensional) Black-Scholes dynamics of the form

$$dS_t = \mathrm{diag}(S_t)\,(\mu dt + AdW_t)$$

with W is a d-dimensional Brownian motion, a vector $\mu \in \mathbb{R}^d$, a volatility matrix $A \in \mathbb{R}^{d\times d}$ and $\operatorname{diag}(S_t)$ is a $d \times d$ diagonal matrix with diagonal components $(S_t^1, \dots, S_t^d)$.

Consider an investor with initial wealth $X_0 = 1$ fully invested in the risky assets using an $\mathbb{R}^d$-valued self-financing strategy $(\pi_t)_{t\geq 0}$, where π_t^i corresponds to the amount of shares held in asset S^i at time t.

General Questions

1. What can you say when $AA^\top = \operatorname{diag}(\sigma_1^2, \dots, \sigma_d^2)$?
2. What is meant by self-financing strategy?
3. Justify that the dynamics of the wealth process of the investor X are given by
$$dX_t = X_t \alpha_t^\top (\mu dt + A dW_t)$$
with $\alpha_t = \operatorname{diag}(S_t)\pi_t / X_t$
4. Interpret the quantity α_t^i.
5. Can you solve the equation for X explicitily? Denote by $\Sigma = AA^\top$ the covariance matrix which we assume to be invertible.

In what follows, we denote X by X^α to make explicit the dependence of the wealth on the investement strategy α. We assume that the investor has a logarithm utility and seeks to maximize $\mathbb{E}[\log X_T^\alpha]$ over all $\mathbb{R}^d$-processes α , where T corresponds to the final time horizon.

Constant case We assume first that the investors looks for investements where α is kept constant trhought the horizon $[0, T]$.

6. For a constant control α what is the distribution of X^α?

7. Show that the optimal deterministic solution $\alpha^{*,det}$ is given by
$$\alpha^{*,det} = \Sigma^{-1}\mu.$$
and compute the optimal value of the problem $J^{*,\mathrm{det}} = \max_{\alpha\in\mathbb{R}^d} \mathbb{E}[\log X_T^\alpha]$

8. Comment.

General case Now we consider general square-integrable processes $(\alpha_t)_{t\geq 0}$.

9. Write the HJB equation for the problem.
10. Making a suitable Ansatz on the solution, solve the problem.
11. Comment.

2. Linear Quadratic control

Let $T > 0$ a finite time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ a filtered probability space.

For a given admissible control in

$$\mathcal{A} = \{\alpha \text{ progressively measurable: } \mathbb{E}[\int_0^T \alpha_s^2]ds < \infty\},$$

we consider the following controlled variable:

$$dX_s^\alpha = (BX_s^\alpha + C\alpha_s)\, ds + (DX_s^\alpha + F\alpha_s)\, dW_s$$

with $B, C, D, F \in \mathbb{R}$ and $X_0 \in \mathbb{R}$.

Our aim is to minimize over all controls in $\mathcal{A}$ the following cost functional:

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left(Q(X_s^\alpha)^2 + N\alpha_s^2\right) ds\right]$$

with $Q \ge 0$ and $N > 0$.

Generalities

1. For $\alpha \in \mathcal{A}$, argue that there exists a unique solution X^α to the Stochastic Differential equation.
2. Can we find X^α explicitely in terms of α?

Probablistic approach

We make the following Ansatz for the value function:

$$V_t^\alpha = \Gamma_t (X^\alpha)_t^2$$

for some deterministic function $t \to \Gamma_t$ to be determined that we assume to be continuously differentiable such that $\Gamma_T = 0$.

Define

$$S_t^\alpha := V_t^\alpha + \int_0^t \left(QX_s^2 + N\alpha_s^2\right) ds$$

3. For $\alpha \in \mathcal{A}$, show that

$$
\begin{aligned}
dS_t^\alpha = X_t^2 &\left(\dot{\Gamma}_t + Q + 2B\Gamma_t + D^2\Gamma_t - \left(N + F^2\Gamma_t\right)^{-1}\left(C\Gamma_t + DF\Gamma_t\right)^2\right) dt \\
&+ \left(N + F^2\Gamma_t\right)\left(\alpha_t - \alpha_t^*\right)^2 dt \\
&+ 2\left(D\Gamma_t X_t^2 + F\alpha_t X_t\right) dW_t
\end{aligned}
$$

for some explicit α^* to be determined. 4. Deduce a necessary and sufficient condition on Γ such that the process $M^\alpha := S^\alpha - \int_0^\cdot \left(N + F^2\Gamma_s\right)(\alpha_s - \alpha_s^*)^2 ds$ is a {local martingale} for any $\alpha \in \mathcal{A}$.

5. Assuming such condition satisfied and assuming that the local martingale M^α is a true martingale show that

$$
J_t(\alpha) - V_t^\alpha \geq 0,
$$

where

$$
J_t(\alpha) := \mathbb{E}\left[\int_t^T \left(QX_s^2 + \alpha_s^2\right) ds | \mathcal{F}_t\right],
$$

for $\alpha \in \mathcal{A}$.

6. Deduce that α^* is an optimal control and $V_t^{\alpha^*}$ is the value function of the problem:

$$
V_t^{\alpha^*} = \inf_{\alpha \in \mathcal{A}_t(\alpha^*)} J_t(\alpha)
$$

where

$$
\mathcal{A}_t(\alpha') := \{\alpha \in \mathcal{A} : \alpha_s = \alpha'_s, \quad s \leq t\}.
$$

PDE approach

7. Solve the problem using the PDE approach.

3. Optimal execution/trading with Market Impact

Let $T > 0$ a finite time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ a filtered probability space.

Consider a trader holding an initial inventory q in some given asset S and is looking to liquidate/trade the asset on interval $[0, T]$. Its inventory at times t is given by:

$$Q_t^u = q - \int_0^t u_s ds$$

with $(u_s)_{0 \leq s \leq t}$ the speed of rate, $t \in [0, T]$ and $q > 0$ the initial amount of shares in the risky asset.

In this model, we suppose that the visible price is composed of three components: the unaffected price and two impacts, one temporary and one transient.

$$S_t = \underbrace{P_t}_{\text{unaffected price}} - \underbrace{\lambda u_t}_{\text{lin. temp}} - \underbrace{\kappa Z_t^u}_{\text{transient}}$$

The transient impact is defined as follows:

$$Z_t^u = \int_0^t G(t-s) u_s ds,$$

for some square integrable kernel G.

In the literature, the transient impact considered are :

- $G(t) = 1$: permanent impact case
- $G(t) = ce^{-\lambda t}$: exponential decay
- $G(t) = c(\epsilon + t)^{-\alpha}$, $\alpha > 0$: power law decay

We introduce the price predictive signal in the unaffected price:

$$P_t = M_t + A_t,$$

$(P_t)_{t \in [0,T]}$ is a semimartingale decomposed in a local martingale $(M_t)_{t \in [0,T]}$ and a finite-variation process $(A_t)_{t \in [0,T]}$. The signal $(A_t)_{t \in [0,T]}$ is defined by:

$$A_t = \int_0^t I_s ds$$

with $(I_t)_{t \in [0,T]}$ a Markovian signal.

Stochastic control problem

The trading cost, affected by the temporary and transient impact is defined by:

$$-\int_0^T S_t dQ_t^u = \int_0^T \left(\left(P_t - \kappa Z_t^u \right) u_t - \lambda u_t^2 \right) dt$$

The trader's terminal wealth is defined as the price of his terminal inventory deducted by the trading costs:

$$Q_T^u P_T - \int_0^T S_t dQ_t^u$$

As we have set terminal constraint on the inventory, we want the terminal inventory to be as close to zero as possible. Thus we introduce a penalty on the terminal inventory given by:

$$-\varrho(Q_T^u)^2$$

We assume that the trader's objective is to liquidate his portfolio by minimizing the trading costs and maximizing the following performance functional:

$$J(u) = \mathbb{E}\left[\int_0^T \left(\left(P_t - \kappa Z_t^u\right) u_t - \lambda u_t^2\right) dt + Q_T^u P_T - \varrho(Q_T^u)^2\right]$$

The goal of this research is to find the optimal strategy u^* solution of our optimal liquidation problem such that:

$$J(u^*) = \sup_{u \in \mathcal{U}} J(u) \tag{1}$$

where $\mathcal{U}$ is a suitable set of admissible trading strategies (controls).

1. Show that the performance functional can be re-written as

$$J(u) = \mathbb{E}\left[\int_0^T \left(\left(A_t - \kappa Z_t^u\right) u_t - \lambda u_t^2\right) dt + A_T Q_T^u - \varrho(Q_T^u)^2\right] + qP_0$$

To simplify the following steps, we drop the term $A_T Q_T^u$ as, at time T, we assume that the trader has liquidated all his portfolio and that the signal has a value close to zero. We decide to drop the term as it is not significant.

From now on, we normalize by 2 the cross term, and consider the functional:

$$J(u) = \mathbb{E}\left[\int_0^T 2\left(\left(A_t - \kappa Z_t^u\right) u_t - \lambda u_t^2\right) dt - \varrho(Q_T^u)^2\right] + qP_0 \tag{2}$$

2. Show that the problem is equivalent to the following control problem in one dimension on X defined by

$$X_t^\alpha = g_0(t) + \int_0^t K(t-s)\left(-\frac{1}{\lambda}X_s^\alpha - \alpha_s\right) ds, \quad g_0(t) = A_t + \varrho q; \tag{3}$$

   with the final performance functional defined by:

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left(-\lambda\alpha_t^2 + \frac{1}{\lambda}(X_t^\alpha)^2\right) dt\right] + qP_0 - \varrho q^2, \tag{4}$$

where we set that the kernel $K(t) = G(t) + \varrho$ and after a suitable change of variables for the control u to α.

3. Assuming that market impact is permanent, i.e. $G(t) = \mathrm{cst}$ and that $A_t = 0$, can you solve the problem?

Bonus: More general kernels and signals

4. What is the difficulty if the market impact kernel G is a general function in L^2, think of power law decay for instance?
5. Can you figue out a way to still solve the problem?

In [ ]:

References 📖

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