

Rough volatility: Part 1

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Main classes of volatility models

Prices are often modeled as continuous semi-martingales of the form

$$dP_t = P_t(\mu_t dt + \sigma_t dW_t).$$

The volatility process σ_s is the most important ingredient of the model. Practitioners consider essentially three classes of volatility models :

- Deterministic volatility (Black and Scholes 1973),
- Local volatility (Dupire 1994),
- Stochastic volatility (Hull and White 1987, Heston 1993, Hagan et al. 2002,...).

In term of regularity, in these models, the volatility is either very smooth or with a smoothness similar to that of a Brownian motion.

Fractional Brownian motion (I)

To allow for a wider range of smoothness, we can consider the fractional Brownian motion in volatility modeling.

Definition

The fractional Brownian motion (fBm) with Hurst parameter H is the only process W^H to satisfy :

- Self-similarity : $(W_{at}^H) \stackrel{\mathcal{L}}{=} a^H(W_t^H)$.
- Stationary increments : $(W_{t+h}^H - W_t^H) \stackrel{\mathcal{L}}{=} (W_h^H)$.
- Gaussian process with $\mathbb{E}[W_1^H] = 0$ and $\mathbb{E}[(W_1^H)^2] = 1$.

Fractional Brownian motion (II)

Proposition

For all $\varepsilon > 0$, W^H is $(H - \varepsilon)$ -Hölder a.s.

Proposition

The absolute moments of the increments of the fBm satisfy

$$\mathbb{E}[|W_{t+h}^H - W_t^H|^q] = K_q h^{Hq}.$$

Fractional Brownian motion

Mandelbrot-van Ness representation

We have

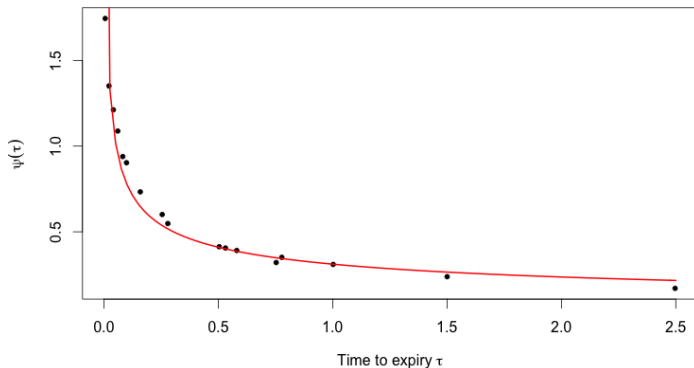
$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}} \right) dW_s.$$

- Classical stochastic volatility models generate reasonable dynamics for the volatility surface.
- However they do not allow to fit the volatility surface, in particular the term structure of the ATM skew :

$$\psi(\tau) := \left| \frac{\partial}{\partial k} \sigma_{\text{BS}}(k, \tau) \right|_{k=0},$$

where k is the log-moneyness and τ the maturity of the option.

About option data : the volatility skew



The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013 ; the red curve is the power-law fit

$$\psi(\tau) = A\tau^{-0.4}.$$

About option data : fractional volatility

- The skew is well-approximated by a power-law function of time to expiry τ . In contrast, conventional stochastic volatility models generate a term structure of ATM skew that is constant for small τ .
- Models where the volatility is driven by a fBm generate an ATM volatility skew of the form $\psi(\tau) \sim \tau^{H-1/2}$, at least for small τ .

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Intraday volatility estimation

We are interested in the dynamics of the (log)-volatility process. We use two proxies for the spot (squared) volatility of a day.

- A 5 minutes-sampling realized variance estimation taken over the whole trading day (8 hours).
- A one hour integrated variance estimator based on the model with uncertainty zones (Robert and R. 2012).

Note that we are not really considering a “spot” volatility but an “integrated” volatility. This might lead to some slight bias in our measurements (which can be controlled).

From now on, we consider realized variance estimations on the S&P over 3500 days, but the results are fairly “universal”.

The log-volatility

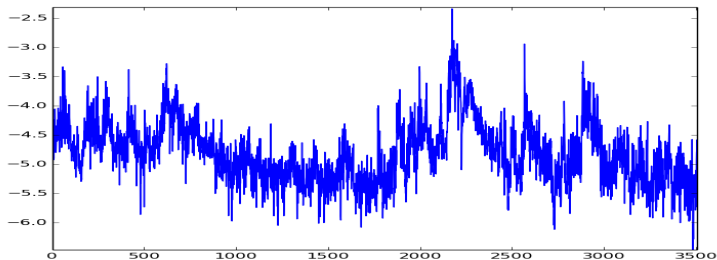


FIGURE – The log volatility $\log(\sigma_t)$ as a function of t , S&P.

Measure of the regularity of the log-volatility

The starting point of this work is to consider the scaling of the moments of the increments of the log-volatility. Thus we study the quantity

$$m(\Delta, q) = \mathbb{E}[|\log(\sigma_{t+\Delta}) - \log(\sigma_t)|^q],$$

or rather its empirical counterpart.

The behavior of $m(\Delta, q)$ when Δ is close to zero is related to the smoothness of the volatility (in the Hölder or even the Besov sense). Essentially, the regularity of the signal measured in l^q norm is s if $m(\Delta, q) \sim c\Delta^{qs}$ as Δ tends to zero.

Scaling of the moments

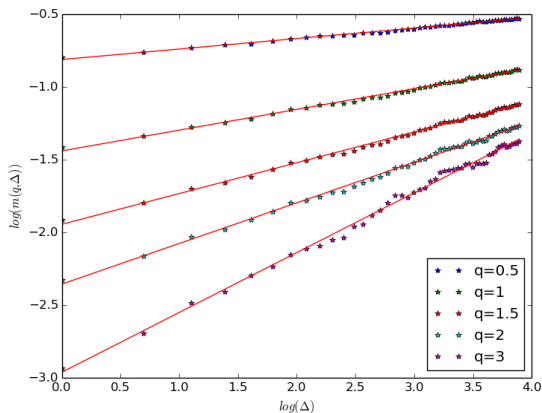


FIGURE $-\log(m(q, \Delta)) = \zeta_q \log(\Delta) + C_q$. The scaling is not only valid as Δ tends to zero, but holds on a wide range of time scales.

Monofractality of the log-volatility

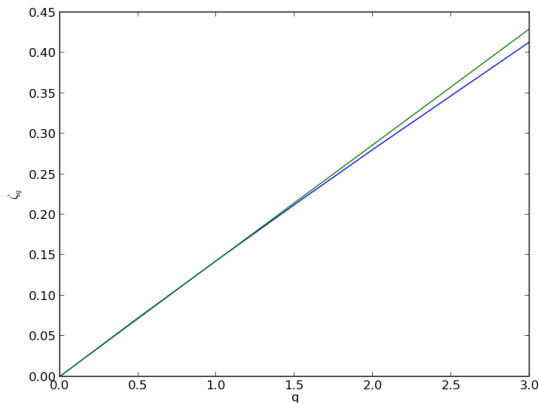


FIGURE – Empirical ζ_q and $q \rightarrow Hq$ with $H = 0.14$ (similar to a fBm with Hurst parameter H).

Distribution of the log-volatility increments

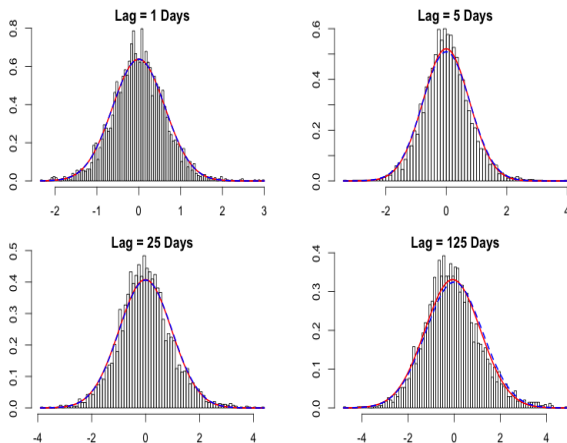


FIGURE – The distribution of the log-volatility increments is close to Gaussian.

A geometric fBm model

The RFSV model

These empirical findings suggest we model the log-volatility as a fractional Brownian motion :

$$\sigma_t = \sigma e^{\nu W_t^H}.$$

- An important property of volatility time series is their multiscaling behavior, see Mantegna and Stanley 2000 and Bouchaud and Potters 2003.
- This means one observes essentially the same law whatever the time scale.
- In particular, there are periods of high and low market activity at different time scales.
- Very few models reproduce this property, see multifractal models.

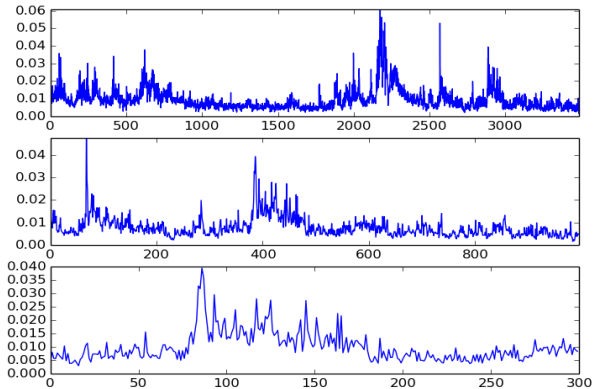


FIGURE – Empirical volatility over 10, 3 and 1 years.

Our model on different time intervals

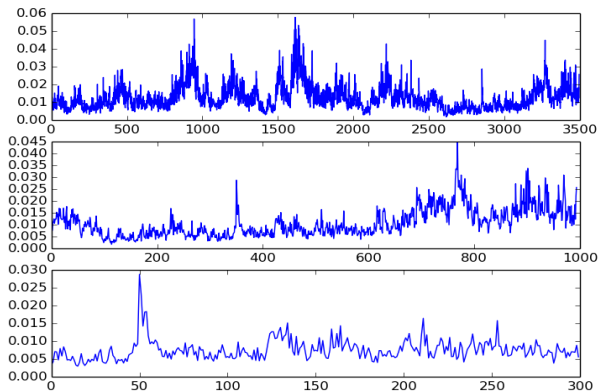


FIGURE – Simulated volatility over 10, 3 and 1 years. We observe the same alternations of periods of high market activity with periods of low market activity.

Apparent multiscaling in our model

- Let $L^{H,\nu}$ be the law on $[0, 1]$ of the process $e^{\nu W_t^H}$.
- Then the law of the volatility process on $[0, T]$ renormalized on $[0, 1]$: σ_{tT}/σ_0 is $L^{H,\nu T^H}$.
- If one observes the volatility on $T = 10$ years (2500 days) instead of $T = 1$ day, the parameter νT^H defining the law of the volatility is only multiplied by $2500^H \sim 3$.
- Therefore, one observes quite the same properties on a very wide range of time scales.
- The roughness of the volatility process ($H = 0.14$) implies a multiscaling behavior of the volatility.

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Prediction of a fractional Brownian motion

There is a nice prediction formula for the fractional Brownian motion.

Proposition (Nuzman and Poor 2000)

For $H < 1/2$

$$\mathbb{E}[W_{t+\Delta}^H | \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \int_{-\infty}^t \frac{W_s^H}{(t-s+\Delta)(t-s)^{H+1/2}} ds.$$

Our prediction formula

We apply the previous formula to the prediction of the log-volatility :

$$\mathbb{E} [\log \sigma_{t+\Delta}^2 | \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \int_{-\infty}^t \frac{\log \sigma_s^2}{(t-s+\Delta)(t-s)^{H+1/2}} ds$$

or more precisely its discrete version :

$$\mathbb{E} [\log \sigma_{t+\Delta}^2 | \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \sum_{k=0}^N \frac{\log \sigma_{t-k}^2}{(k+\Delta+1/2)(k+1/2)^{H+1/2}}.$$

We compare it to usual predictors using the criterion

$$P = \frac{\sum_{k=1}^{N-\Delta} (\widehat{\log(\sigma_{k+\Delta}^2)} - \log(\sigma_{k+\Delta}^2))^2}{\sum_{k=1}^{N-\Delta} (\log(\sigma_{k+\Delta}^2) - \mathbb{E}[\log(\sigma_{k+\Delta}^2)])^2}.$$

	AR(5)	AR(10)	HAR(3)	RFSV
SPX2.rv $\Delta = 1$	0.317	0.318	0.314	0.313
SPX2.rv $\Delta = 5$	0.459	0.449	0.437	0.426
SPX2.rv $\Delta = 20$	0.764	0.694	0.656	0.606
FTSE2.rv $\Delta = 1$	0.230	0.229	0.225	0.223
FTSE2.rv $\Delta = 5$	0.357	0.344	0.337	0.320
FTSE2.rv $\Delta = 20$	0.651	0.571	0.541	0.472
N2252.rv $\Delta = 1$	0.357	0.358	0.351	0.345
N2252.rv $\Delta = 5$	0.553	0.533	0.513	0.504
N2252.rv $\Delta = 20$	0.875	0.795	0.746	0.714
GDAXI2.rv $\Delta = 1$	0.237	0.238	0.234	0.231
GDAXI2.rv $\Delta = 5$	0.372	0.362	0.350	0.339
GDAXI2.rv $\Delta = 20$	0.661	0.590	0.550	0.498
FCHI2.rv $\Delta = 1$	0.244	0.244	0.241	0.238
FCHI2.rv $\Delta = 5$	0.378	0.373	0.366	0.350
FCHI2.rv $\Delta = 20$	0.669	0.613	0.598	0.522

Regression window and horizon

After a simple change of variable, the prediction of the log-volatility can be written :

$$\mathbb{E}[\log(\sigma_{t+\Delta}^2)|\mathcal{F}_t] \sim \frac{\cos(H\pi)}{\pi} \int_0^1 \frac{\log(\sigma_{t-\Delta u}^2)}{(u+1)u^{H+1/2}} du.$$

The only time scale that appears in the above regression is the horizon Δ .

As it is known by practitioners :

If trying to predict volatility one week ahead, one should essentially look at the volatility over the last week. If trying to predict the volatility one month ahead, one should essentially look at the volatility over the last month.

Conditional distribution of the fractional Brownian motion and prediction of the variance

Proposition (Nuzman and Poor 2000)

In law,

$$W_{t+\Delta}^H | \mathcal{F}_t = \mathcal{N}(\mathbb{E}[W_{t+\Delta}^H | \mathcal{F}_t], c\Delta^{2H})$$

with

$$c = \frac{\sin(\pi(1/2 - H))\Gamma(3/2 - H)^2}{\pi(1/2 - H)\Gamma(2 - 2H)}.$$

Therefore, our predictor of the variance writes :

$$\mathbb{E}[\sigma_{t+\Delta}^2 | \mathcal{F}_t] = e^{\mathbb{E}[\log(\sigma_{t+\Delta}^2) | \mathcal{F}_t] + 2\nu^2 c \Delta^{2H}}.$$

	AR(5)	AR(10)	HAR(3)	RFSV
SPX2.rv $\Delta = 1$	0.520	0.566	0.489	0.475
SPX2.rv $\Delta = 5$	0.750	0.745	0.723	0.672
SPX2.rv $\Delta = 20$	1.070	1.010	1.036	0.903
FTSE2.rv $\Delta = 1$	0.612	0.621	0.582	0.567
FTSE2.rv $\Delta = 5$	0.797	0.770	0.756	0.707
FTSE2.rv $\Delta = 20$	1.046	0.984	0.935	0.874
N2252.rv $\Delta = 1$	0.554	0.579	0.504	0.505
N2252.rv $\Delta = 5$	0.857	0.807	0.761	0.729
N2252.rv $\Delta = 20$	1.097	1.046	1.011	0.964
GDAXI2.rv $\Delta = 1$	0.439	0.448	0.399	0.386
GDAXI2.rv $\Delta = 5$	0.675	0.650	0.616	0.566
GDAXI2.rv $\Delta = 20$	0.931	0.850	0.816	0.746
FCHI2.rv $\Delta = 1$	0.533	0.542	0.470	0.465
FCHI2.rv $\Delta = 5$	0.705	0.707	0.691	0.631
FCHI2.rv $\Delta = 20$	0.982	0.952	0.912	0.828