# Supremum of a diffusion bridge

## Proposition

The distribution of the supremum of the Brownian bridge starting at 0 and arriving at $y$ at time $T$, defined by $Y_t^{W,T,y} = \frac{t}{T}y + W_t - \frac{t}{T}W_T$ on $[0,T]$, is given by

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Y_t^{W,T,y} \leqslant z\Big) = \begin{cases} 1 - \exp\left(-\frac{2}{T}z(z-y)\right) & \text{if } z \geqslant \max(y,0), \\ 0 & \text{if } z \leqslant \max(y,0). \end{cases}$$

The key of the proof is the reflection principle.

# Reflection principle

The key of the proof is the reflection principle which says that, if $m > 0$ and

$$\tau_m = \inf\{t : W_t \geqslant m\} = \inf\{t : W_t = m\}$$

then

$$(2m - W_{\tau_a+t})_{t\geqslant 0} = \big(a - (W_{\tau_m+t} - W_{\tau_a})\big)_{t\geqslant 0} \text{ is a standard B.M.}$$
$$\text{starting from } m \text{ and } \perp\!\!\!\perp \text{ of } W^{\tau_m} = (W_{\tau_m\wedge t})_{t\geqslant 0}.$$

Then [. . . ], for every $m \geqslant \max(w, 0)$,

$$\mathbb{P}\Big(\sup_{t\in[0,T]} W_t \geqslant m,\ W_T \leqslant w\Big) = \mathbb{P}(W_T \geqslant 2m - w).$$

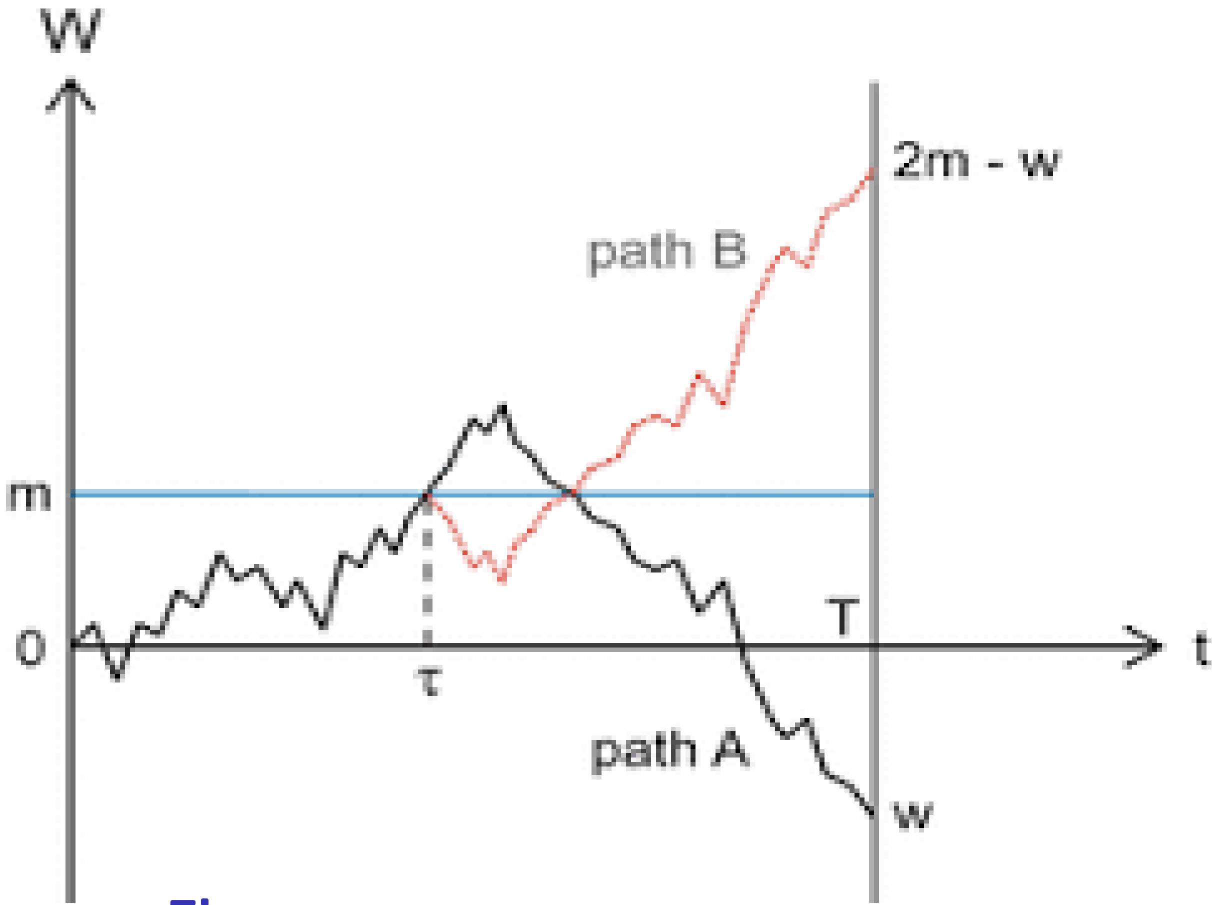


**Figure:** Reflection principle for the Brownian motion.

# Proof

- Have in mind

$$\mathcal{L}\big(Y^{W,T,y}\big) = \mathcal{L}\big(W_{t\in[0,T]}\,|\,W_T = y\big).$$

- Hence, we can derive the result from an expression of the joint distribution of $\big(\sup_{t\in[0,T]} W_t, W_T\big)$,
- characterized by

$$\mathbb{P}\Big(\sup_{t\in[0,T]} W_t \geqslant z,\ W_T \leqslant y\Big)\cdots = \mathbb{P}(W_T \geqslant 2z-y),\ z\geqslant \max(y,0).$$

- Let us try! For every $z \geqslant \max(y,0)$,

$$\mathbb{P}\Big(\sup_{t\in[0,T]} W_t \geqslant z,\ W_T \leqslant y\Big) = \int_{2z-y}^{+\infty} \frac{e^{-\frac{\xi^2}{2T}}}{\sqrt{2\pi T}}\,d\xi.$$

- Hence, since the involved functions are differentiable one has

$$\mathbb{P}(\sup_{t\in[0,T]} W_t \geqslant z \,|\, W_T = y) = \lim_{\eta\to 0} \frac{\Big(\mathbb{P}(W_T \geqslant 2z-(y+\eta)) - \mathbb{P}(W_T \geqslant 2z-y)\Big)/\eta}{\Big(\mathbb{P}(W_T \leqslant y+\eta) - \mathbb{P}(W_T \leqslant y)\Big)/\eta}$$

- so that

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,T]} W_t \geqslant z \,|\, W_T = y) &= \frac{\partial_y \mathbb{P}(W_T \geqslant 2z-y)}{\partial_y \mathbb{P}(W_T \leqslant y)}\\ &= \frac{e^{-\frac{(2z-y)^2}{2T}}}{e^{-\frac{y^2}{2T}}}\\ &= e^{-\frac{(2z-y)^2-y^2}{2T}}\\ &= e^{-\frac{2z(z-y)}{2T}}.\end{aligned}$$

$\square$

## Corollary

Let $\lambda > 0$ and let $x, y \in \mathbb{R}$. If $Y^{W,T}$ denotes the standard Brownian bridge of $W$ between 0 and $T$, then for every $z \in \mathbb{R}$,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\Big(x+(y-x)\frac{t}{T}+\lambda Y_t^{W,T}\Big)\leqslant z\right)=\begin{cases}1-e^{\left(-\frac{2}{T\lambda^2}(z-x)(z-y)\right)} & \text{if } z\geqslant \max(x,y),\\ 0 & \text{if } z<\max(x,y).\end{cases}$$

*Proof.*

- First note that

$$x+(y-x)\frac{t}{T}+\lambda Y_t^{W,T}=\lambda x'+\lambda\left(y'\frac{t}{T}+Y_t^{W,T}\right) \text{ with } x'=x/\lambda,\ y'=(y-x)/\lambda.$$

- Then use the above proposition and that for a r.v. $\xi$, $\alpha\in\mathbb{R}$, $\beta\in(0,+\infty)$,

$$\mathbb{P}(\alpha+\beta\,\xi\leqslant z)=\mathbb{P}\Big(\xi\leqslant\frac{z-\alpha}{\beta}\Big)=1-\mathbb{P}\Big(\xi>\frac{z-\alpha}{\beta}\Big). \qquad \square$$

**Exercise.** Show using $-W \stackrel{d}{=} W$ that

$$\mathbb{P}\left(\inf_{t\in[0,T]}(x+(y-x)\frac{t}{T}+\lambda Y_t^{W,T})\leqslant z\right)=\begin{cases}\exp\left(-\frac{2}{T\lambda^2}(z-x)(z-y)\right) & \text{if } z\leqslant\min(x,y),\\ 1 & \text{if } z>\min(x,y).\end{cases}$$

# Application to Lookback like options (1)

- We want to compute (an approximation of)

$$e^{-rT}\mathbb{E}\, f\big(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t\big)$$

by a Monte Carlo simulation based on the genuine Euler scheme.

- We first note, owing to the tower rule for conditional expectation, that

$$\mathbb{E}\, f\big(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t\big) = \mathbb{E}\Big[\mathbb{E}\big(f(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t)\,|\,\bar{X}^n_{t_\ell}, \ell=0,\ldots,n\big)\Big].$$

- We derive from the former theorems that

$$\mathbb{E}\,\big(f(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t)\,|\,\bar{X}^n_{t_\ell}=x_\ell, \ell=0,\ldots,n\big) = f\big(x_n, \max_{0\leqslant k\leqslant n-1} M^{n,k}_{x_k,x_{k+1}}\big)$$

where the random variables

$$M^{n,k}_{x_k,x_{k+1}} := \sup_{t\in[0,T/n]}\Big(x_k + \frac{nt}{T}(x_{k+1}-x_k) + \sigma(t^n_k, x_k)\, Y^{W^{(t^n_k)},T/n}_t\Big)$$

are independent (as supremums of independent Brownian bridges).

# Application to Lookback like option (2)

- Following the above theorem, the c.d.f. function $G^{n,k}_{x,y}$ of $M^{n,k}_{x,y}$ reads

$$G^{n,k}_{x,y}(z) = \left[1 - \exp\left(-\frac{2n}{T\sigma^2(t^n_k, x)}(z-x)(z-y)\right)\right]\mathbf{1}_{\left\{z\geqslant \max(x,y)\right\}}.$$

- The inverse distribution simulation rule yields that

$$\sup_{t\in[0,T/n]}\left(x + \frac{t}{\frac{T}{n}}(y-x) + \sigma(t^n_k, x)Y_t^{W^{(t^n_k)},T/n}\right) \stackrel{d}{=} (G^{n,k}_{x,y})^{-1}(U),$$
$$\stackrel{d}{=} (G^{n,k}_{x,y})^{-1}(1-U),$$

where $U \stackrel{d}{=} \mathcal{U}([0,1]$ (so that $U \stackrel{d}{=} 1-U$).

- Determining $(G^{n,k}_{x,y})^{-1}$ at $1-u$ amounts to solving the equation

$$G^{n,k}_{x,y}(z) := 1-u \quad \text{under the constraint } z \geqslant \max(x,y),$$

$$i.e. \quad 1 - \exp\left(-\frac{2n}{T\sigma^2(t^n_k, x)}(z-x)(z-y)\right) = 1-u,\ z\geqslant \max(x,y).$$

# Application to Lookback like option (3)

- Or, equivalently,

$$z^2 - (x+y)z + xy + \frac{T}{2n}\sigma^2(t_k^n, x)\log(u) = 0, \quad z \geqslant \max(x,y).$$

- The above equation has two solutions (with $\pm$), only the solution below satisfies the constraint. Consequently,

$$(G_{x,y}^{n,k})^{-1}(1-u) = \frac{1}{2}\left(x + y + \sqrt{(x-y)^2 - 2T\sigma^2(t_k^n, x)\log(u)/n}\right).$$

$$\geqslant \tfrac{1}{2}(x+y+|x-y|) = \max(x,y) \quad \textbf{since } \log u < \mathbf{0}.$$

- Finally,

$$\mathcal{L}\Big(\max_{t\in[0,T]} \bar{X}_t^n \,|\, \bar{X}_{t_k}^n = x_k,\ k = 0,\ldots,n\Big) = \mathcal{L}\Big(\max_{0\leqslant k\leqslant n-1} (G_{x_k,x_{k+1}}^{n,k})^{-1}(U_{k+1})\Big),$$

where $(U_k)_{1\leqslant k\leqslant n}$ are i.i.d.with $U([0,1])$-distribution.

# Pseudo-code for Lookback style options ($r = 0$)

- By Lookback style options we mean the class of options whose payoff involve possibly both $\bar{X}^n_T$ and the maximum of $(X_t)$ over $[0, T]$, *i.e.*

$$\mathbb{E}\, f\big(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t\big).$$

Examples. *Regular Call on maximum* $\rightsquigarrow f(x, y) = (y - K)_+$; *(maximum) Lookback option* $\rightsquigarrow f(x, y) = y - x$; *(maximum) partial lookback* $\rightsquigarrow f_\lambda(x, y) = (y - \lambda x)_+$, $\lambda > 1$.

- MC approximation of $\mathbb{E}\, f\big(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t\big)$ using the genuine Euler scheme.

• $S^f = 0$.

for $m = 1 : M$
  - Simulate a path of the discrete time Euler scheme and set $x_k := \bar{X}^{n,(m)}_{t^n_k}$, $k = 0, \ldots, n$.
  - Simulate $\Xi^{(m)} := \max_{0\leqslant k\leqslant n-1}\big(G^{n,k}_{x_k,x_{k+1}}\big)^{-1}(U^{(m)}_k)$, $(U^{(m)}_k)_{1\leqslant k\leqslant n}$ i.i.d. $\sim \mathcal{U}([0,1])$.
  - Compute $f(\bar{X}^{n,(m)}_T, \Xi^{(m)})$.
  - Compute $S^f_m := f(\bar{X}^{n,(m)}_T, \Xi^{(m)}) + S^f_{m-1}$.

end.$(m)$

• $\mathbb{E}\, f\big(\bar{X}^n_T, \sup_{t\in[0,T]} \bar{X}^n_t\big) \simeq \frac{S^f_M}{M}$ [1].

[1] . . . Of course one needs to compute the empirical variance (approximately) in order to design a confidence interval, without which the method is simply nonsense. . . .

# Exercise and WARNING!

- $(a)$ Show that the distribution of the infimum of the Brownian bridge $(Y_t^{W,T,y})_{t\in[0,T]}$ starting at 0 and arriving at $y$ at time $T$ is given by

$$\mathbb{P}\Big(\inf_{t\in[0,T]} Y_t^{W,T,y} \leqslant z\Big) = \begin{cases} \exp\left(-\frac{2}{T}z(z-y)\right) & \text{if } z \leqslant \min(y,0),\\ 1 & \text{if } z \geqslant \min(y,0).\end{cases}$$

- $(b)$ Derive a formula for the conditional distribution of the minimum of the continuous Euler scheme using now the inverse distribution functions

$$(F_{x,y}^{n,k})^{-1}(u) = \frac{1}{2}\left(x+y-\sqrt{(x-y)^2-2\,T\sigma^2(t_k^n,x)\log(u)/n}\right),\ \ u\in(0,1),$$

of the random variable $\displaystyle \inf_{t\in[0,T/n]}\left(x+\frac{t}{\frac{T}{n}}(y-x)+\sigma(t_k^n,x)Y_t^{W,T/n}\right).$

- **WARNING!** The above method is not appropriate (in fact pure nonsense) for simulating the joint distribution of the $(n+3)$-tuple $\big(\bar{X}^n_{t_k^n},\ k=0,\ldots,n, \inf_{t\in[0,T]}\bar{X}_t^n, \sup_{t\in[0,T]}\bar{X}_t^n\big)$.

# Application to regular barrier options

- Up-and-Out Call option premium: $\mathbb{E}\left(f(X_T)\mathbf{1}_{\{\sup_{t\in[0,T]} X_t\leqslant L\}}\right)$
- Remember variance reduction by pre-conditioning (Blackwell-Rao)!
- Let us derive a general weighted (semi-closed) formula for $\mathbb{E}\left(f(\bar{X}^n_T)\mathbf{1}_{\{\sup_{t\in[0,T]} \bar{X}^n_t\leqslant L\}}\right)$ (genuine Euler scheme).

## Proposition (Up-and-Out Call option)

$$\mathbb{E}\Big(f(\bar{X}^n_T)\mathbf{1}_{\big\{\sup_{t\in[0,T]} \bar{X}^n_t\leqslant L\big\}}\Big)$$
$$= \mathbb{E}\left[f(\bar{X}^n_T)\mathbf{1}_{\{\max_{0\leqslant k\leqslant n} \bar{X}_{t_k}\leqslant L\}}\prod_{k=0}^{n-1}\left(1-e^{-\frac{2n}{T}\frac{(\bar{X}^n_{t_k}-L)(\bar{X}_{t_{k+1}}-L)}{\sigma^2(t^n_k,X_{t_k})}}\right)\right].$$

# Comment

- Furthermore, we know that *the random variable in the right-hand side always has a lower variance* owing to pre-conditioning

$$\mathrm{Var}\left(f(\bar X^n_T)\mathbf{1}_{\left\{\max_{0\leqslant k\leqslant n}\bar X^n_{t_k}\leqslant L\right\}}\prod_{k=0}^{n-1}\left(1-e^{-\frac{2n}{T}\frac{(\bar X^n_{t_k}-L)(\bar X^n_{t_{k+1}}-L)}{\sigma^2(t^n_k,\bar X^n_{t_k})}}\right)\right)$$
$$\leqslant \mathrm{Var}\left(f(\bar X^n_T)\mathbf{1}_{\left\{\sup_{t\in[0,T]}\bar X^n_t\leqslant L\right\}}\right).$$

- Conclusion: smaller bias and lower variance: win-win procedure !

# Proof

- We start from the chaining rule for conditional expectations:

$$\mathbb{E}\left(f(\bar{X}^n_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}^n_t\leqslant L\}}\right)=\mathbb{E}\left[\mathbb{E}\Big(f(\bar{X}^n_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}^n_t\leqslant L\}}\,|\,\bar{X}^n_{t_k},\;k=0,\ldots,n\Big)\right].$$

- Then use the conditional independence of the bridges of the genuine Euler scheme given $\bar{X}^n_{t_k}$, $k=0,\ldots,n$:

$$\begin{aligned}\mathbb{E}\Big(f(\bar{X}^n_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}^n_t\leqslant L\}}\Big)&=\mathbb{E}\left(f(\bar{X}^n_T)\mathbb{P}\Big(\sup_{t\in[0,T]}\bar{X}^n_t\leqslant L\,|\,\bar{X}_{t_k},\;k=0,\ldots,n\Big)\right)\\&=\mathbb{E}\left(f(\bar{X}^n_T)\prod_{k=1}^{n}G_{\bar{X}^n_{t_k},\bar{X}^n_{t_{k+1}}}(L)\right)\\&=\mathbb{E}\left[f(\bar{X}^n_T)\mathbf{1}_{\left\{\max_{0\leqslant k\leqslant n}\bar{X}^n_{t_k}\leqslant L\right\}}\prod_{k=0}^{n-1}\left(1-e^{-\frac{2n}{T}\frac{(\bar{X}^n_{t_k}-L)(\bar{X}^n_{t_{k+1}}-L)}{\sigma^2(t^n_k,\bar{X}^n_{t_k})}}\right)\right].\qquad\square\end{aligned}$$

# Exercises

1. *Down-and-Out option.* Show likewise that for every Borel function $f \in L^1(\mathbb{R}, \mathbb{P}_{\bar{X}^n_T})$,

$$\mathbb{E}\left(f(\bar{X}^n_T)\ \mathbf{1}_{\left\{\inf_{t\in[0,T]}\bar{X}^n_t \geqslant L\right\}}\right) = \mathbb{E}\left(f(\bar{X}^n_T)\mathbf{1}_{\left\{\min_{0\leqslant k\leqslant n}\bar{X}^n_{t_k} \geqslant L\right\}} e^{-\frac{2n}{T}\sum_{k=0}^{n-1}\frac{(\bar{X}^n_{t_k}-L)(\bar{X}^n_{t_{k+1}}-L)}{\sigma^2(t^n_k,\bar{X}^n_{t_k})}}\right)$$

and that the expression in the second expectation has a lower variance.

2. Extend the above results to Parisian options i.e. to varying barriers of the form

$$L(t) := e^{at+b},\ a,\ b \in \mathbb{R}.$$

# Back to sensitivity computation: framework

- Alternative terminology: $\delta$-hedging, Greeks computation.
- Let $(E, \mathcal{E})$ be a measurable space ($E = \mathbb{R}^d$, $E = \mathcal{C}([0, T], \mathbb{R}^d)$, etc). Let $I$ a non-trivial interval.
- Let $F : I \times E \to \mathbb{R}$ be $\mathcal{B}or(I) \otimes \mathcal{E}$-measurable and let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E})$.

$$f(x) = \mathbb{E}\, F(x, Z).$$

- Alternative: $f(x) = \mathbb{E}_\omega\, F(x, \omega)$. But $Z$ is convenient.
- Assume $f$ is regular at some point(s).
- How to compute $f'(x)$?
- Several methods.

# What we (already) know

- With $F : I \times \Omega \to \mathbb{R}$ i.e. $F(x,\omega)$

(*i*) $x \longmapsto F(x, Z(\omega))$ is $\mathbb{P}(d\omega)$-*a.s.* diff. at $x_0 \in I$

(*ii*) The family $\left(\dfrac{F(x,\omega)-F(x_0,\omega)}{x-x_0}\right)_{x\in\mathcal{V}(x_0)\setminus\{x_0\}}$ is ($L^1$-)uniformly integrable, [Which holds whenever the dame family if $L^p$-boubnded for some $p>1$]

- Then

$$f'(x) = \mathbb{E}\left(\partial_x F(x,\omega)\right)$$

- Hence recent come back of (Adjoint) Automatic differentiation.
- Difficulty to compute $\partial_x F(x,\omega)$ (and laziness) led to Bump method...

# Finite difference (or Bump/shock) method(s)

## Proposition

Let $x \in \mathbb{R}$.

- Assume $F$ satisfies the local mean quadratic Lipschitz assumption at $x$

$$\exists\, \varepsilon_0 > 0,\ \forall\, x' \in (x - \varepsilon_0, x + \varepsilon_0), \quad \big\| F(x, Z) - F(x', Z) \big\|_2 \leqslant C_{F,Z} |x - x'|.$$

- Assume $f$ is twice differentiable with $f''$ Lipschitz on $(x - \varepsilon_0, x + \varepsilon_0)$.
- Let $(Z_k)_{k \geqslant 1}$ be i.i.d. r.v. $\sim \mathcal{L}(Z)$.
- Then for every $\varepsilon \in (0, \varepsilon_0)$, the mean quadratic (*RMSE*) error satisfies

$$\left\| f'(x) - \frac{1}{M} \sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon} \right\|_2 \leqslant \sqrt{\left( [f'']_{\rm Lip} \frac{\varepsilon^2}{2} \right)^2 + \frac{C_{F,Z}^2 - \left(f'(x) - \frac{\varepsilon^2}{2}[f'']_{\rm Lip}\right)_+^2}{M}}$$

$$\leqslant \sqrt{\left( [f'']_{\rm Lip} \frac{\varepsilon^2}{2} \right)^2 + \frac{C_{F,Z}^2}{M}} \leqslant [f'']_{\rm Lip} \frac{\varepsilon^2}{2} + \frac{C_{F,Z}}{\sqrt{M}}.$$

If $f^{(3)}$ exists on $(x - \varepsilon_0, x + \varepsilon_0)$ and bounded, then $[f'']_{\rm Lip} \rightsquigarrow \frac{1}{3} \sup_{|\xi - x| \leqslant \varepsilon_0} |f^{(3)}(\xi)|$.

- $[f'']_{\rm Lip}\frac{\varepsilon^2}{2}$ represents the *bias*
- $\frac{C_{F,Z}}{\sqrt{M}}$ the *statistical* error (due to the *SLLN*).

# Proof

- Let $\varepsilon \in (0, \varepsilon_0)$. Taylor formula applied to $f$ between $x$ and $x \pm \varepsilon$ yields

$$\left| f'(x) - \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right| \leqslant [f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2}.$$

- The structural bias reads

$$\mathbb{E}\left(\frac{F(x+\varepsilon, Z) - F(x-\varepsilon, Z)}{2\varepsilon}\right) = \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon}$$

and the structural variance reads

$$\begin{aligned}
\mathrm{Var}\left(\frac{F(x+\varepsilon, Z) - F(x-\varepsilon, Z)}{2\varepsilon}\right) &= \mathbb{E}\left(\left(\frac{F(x+\varepsilon, Z) - F(x-\varepsilon, Z)}{2\varepsilon}\right)^2\right) - \left(\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon}\right)^2 \\
&= \frac{\|F(x+\varepsilon, Z) - F(x-\varepsilon, Z)\|_2^2}{4\varepsilon^2} - \left(\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon}\right)^2 \\
&\leqslant C_{F,Z}^2 \frac{(2\varepsilon)^2}{4\varepsilon^2} - \left(\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon}\right)^2 \\
&= C_{F,Z}^2 - \left(\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon}\right)^2 \\
&\leqslant C_{F,Z}^2.
\end{aligned}$$

- Using the bias-variance decomposition for random varaible $Y \in L^2(\mathbb{P})$,

$$\mathbb{E}\,(a - Y)^2 = (a - \mathbb{E}\,Y)^2 + \mathrm{Var}(Y).$$

- We derive the following upper-bound for the mean squared error

$$\begin{aligned}
&\left\| f'(x) - \frac{1}{M}\sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon} \right\|_2^2 \\
&= \left( f'(x) - \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)^2 \\
&\quad + \mathrm{Var}\left( \frac{1}{M}\sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon} \right) \\
&= \left( f'(x) - \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)^2 \\
&\quad + \frac{1}{M}\mathrm{Var}\left( \frac{F(x+\varepsilon, Z) - F(x-\varepsilon, Z)}{2\varepsilon} \right) \\
&\leqslant \left( [f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2} \right)^2 + \frac{1}{M}\left( C_{F,Z}^2 - \left( \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)^2 \right)
\end{aligned}$$

where we combined the former bounds.

- To get the improved bound we use that

$$\frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon} \geqslant f'(x) - \frac{\varepsilon^2}{2}[f'']_{\mathrm{Lip}}$$

so that

$$\begin{aligned}
\left|\mathbb{E}\left(\frac{F(x+\varepsilon,Z)-F(x-\varepsilon,Z)}{2\varepsilon}\right)\right| &= \left|\frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon}\right| \\
&\geqslant \left(\frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon}\right)_+ \\
&\geqslant \left(f'(x) - \frac{\varepsilon^2}{2}[f'']_{\mathrm{Lip}}\right)_+ .
\end{aligned}$$

- Then plug this lower bound into the “definition” of the variance yields the announced inequality. □

# ℵ Practitioner's corner

- The above result suggests to equalize the variance and the statistical error term by setting

$$M = M(\varepsilon) = \left(\frac{2C_{F,Z}}{\varepsilon^2[f'']_{\rm Lip}}\right)^2 = \left(\frac{2C_{F,Z}}{[f'']_{\rm Lip}}\right)^2 \varepsilon^{-4}.$$

- Not realistic: $C_{F,Z}$ is only an upper-bound of the variance term and $[f'']_{\rm Lip}$ is usually unknown.
- An alternative is to choose

$$M(\varepsilon) = o\big(\varepsilon^{-4}\big) \quad \text{e.g.} \quad O(\varepsilon^{-4}/\log(1/\varepsilon))$$

  to make the bias more and more negligible w.r.t the variance when $\varepsilon \to 0$
- Practical approach: have in mind that reducing the *RMSE* error by a 2-factor needs to reduce $\varepsilon$ and increase $M$ as follows:

$$\varepsilon \rightsquigarrow \varepsilon/\sqrt{2} \qquad \text{and} \qquad M \rightsquigarrow 4\,M.$$

# Warning (what should never be done)!

- If using two *independent* samples $(Z_k)_{k\geqslant 1}$ and $(\widetilde{Z}_k)_{k\geqslant 1}$ of teh ditribution of $Z$ to simulate copies $F(x-\varepsilon, Z)$ and $F(x+\varepsilon, Z)$.
- Then,

$$\begin{aligned}\mathrm{Var}\Big(\frac{1}{M}\sum_{k=1}^{M}\frac{F(x+\varepsilon,Z_k)-F(x-\varepsilon,\widetilde{Z}_k)}{2\varepsilon}\Big)\\ &= \frac{1}{4M\varepsilon^2}\Big(\mathrm{Var}\big(F(x+\varepsilon,Z)\big)+\mathrm{Var}\big(F(x-\varepsilon,Z)\big)\Big)\\ &\simeq \frac{\mathrm{Var}\big(F(x,Z)\big)}{2M\varepsilon^2}.\end{aligned}$$

- The asymptotic variance of the estimator explodes as $\varepsilon \to 0$ so that

$$RMSE \simeq [f'']_{\mathrm{Lip}}\frac{\varepsilon^2}{2}+\frac{\mathrm{Std}\big(F(x,Z)\big)}{\varepsilon\sqrt{2M}},$$

- Leads to unrealistic constraint $M(\varepsilon) \propto \varepsilon^{-6}$ to keep the balance between the bias and variance term (one hundred times more simulations!).
- Equivalently to switch $\varepsilon \rightsquigarrow \varepsilon/\sqrt{2}$ and $M \rightsquigarrow 8\,M$ to reduce the error by a 2-factor.

# Examples (Greeks computation) I.

- *Sensitivity in a Black–Scholes model.*
  - Vanilla payoffs $g : \mathbb{R}_+ \to \mathbb{R}$ viewed as functions of $\mathcal{N}(0;1)$ distribution

$$F(x,z) = e^{-rT} g\left(x\, e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}\right),\ z\in\mathbb{R},\ x\in(0,+\infty),$$

  - If $g$ is Lipschitz continuous, then

$$|F(x,Z)-F(x',Z)| \leqslant [g]_{\rm Lip}|x-x'|e^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}Z}$$

  - Elementary computations show, using that $Z \stackrel{d}{=} \mathcal{N}(0;1)$,

$$\|F(x,Z)-F(x',Z)\|_2 \leqslant [h]_{\rm Lip}|x-x'|e^{\frac{\sigma^2}{2}T}.$$

  - Regularity of $f$ follows from an easy change of variable ($\mu = r-\frac{\sigma^2}{2}$)

$$f(x) = e^{-rT}\int_{\mathbb{R}} h\left(x\, e^{\mu T+\sigma\sqrt{T}z}\right)e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} = e^{-rT}\int_0^{+\infty} h(y)e^{-\frac{(\log(x/y)+\mu T)^2}{2\sigma^2 T}}\frac{dy}{y\sigma\sqrt{2\pi T}}$$

    The second integral appears as a $\frac{dy}{y}$-convolution on $(0,+\infty)$.
  - Hence $f$ is infinitely differentiable on $(0,+\infty)$.
  - So that $f^{(k)}$ are loc-Lipschitz Lipschitz on $(0,+\infty)$.

# Examples (Greeks computation) II.

- *Diffusion model with Lipschitz continuous coefficients.* Let $X^x = (X_t^x)_{t\in[0,T]}$ denote the Brownian diffusion solution of

$$(SDE) \;\equiv\; dX_t = b(X_t)dt + \vartheta(X_t)dW_t, \quad X_0 = x$$

  where $b$ and $\vartheta$ are Lipschitz.
- We should instead write $F(x,\omega) = h\big(X_T^x(\omega)\big)$.
- Lipschitz continuity of the flow of the above $SDE$ yields

$$\big\|F(x,\,.\,) - F(x',\,.\,)\big\|_2 \leqslant C_{b,\vartheta}[h]_{\rm Lip}|x - x'|e^{C_{b,\vartheta}T} \tag{2}$$

  where $C_{b,\vartheta}$ is a positive constant only depending on the Lipschitz coefficients of $b$ and $\vartheta$.
- Also holds for $\|\cdot\|_{\sup}$-Lip path-dependent functionals.
- Regularity of $f(x) = \mathbb{E}\,F(x,\omega)$ is less straightforward...
- Holds in two situations: $\triangleright$ $h$, $b$ and $\sigma$ are regular enough to apply pathwise differentiation of $x \mapsto F(x,\omega)$
  $\triangleright$ or $\vartheta$ satisfies a uniform ellipticity assumption $\vartheta \geqslant \varepsilon_0 > 0$.

# Examples (Greeks computation) III.

- *Euler scheme of a Brownian diffusion model with Lipschitz continuous coefficients.* The same holds for the Euler scheme. Furthermore, Assumption (2) holds uniformly with respect to $n$ if $\frac{T}{n}$ is the step size of the Euler scheme.
- $F$ can also be a functional of the whole path of a diffusion, provided $F$ is Lipschitz continuous with respect to the sup-norm over $[0, T]$ i.e. is $\|\cdot\|_{\sup}$-Lipschitz functional from $C([0, T], \mathbb{R}^d) to \mathbb{R}$.

# Singular setting

- Above setting often testifies of laziness (no excuse now with *AAD*. . . )
- But in the case of *singular setting*: $f$ is differentiable at $x$ but $F(\,.\,,Z)$ is not $L^2$-Lipschitz

## Proposition (Singular setting)

Let $x \in \mathbb{R}$.

- Assume that $F$ satisfies an $L^2(\mathbb{P})$ $\theta$-Hölder assumption, $\theta \in (0,1]$ in the neighbourhood of $x$: there exists a positive real constant $C_{Hol,F,Z}$

$$\forall\, x', x'' \in (x-\varepsilon_0, x+\varepsilon_0), \quad \big\|F(x',Z)-F(x'',Z)\big\|_2 \leqslant C_{Hol,F,Z}|x'-x''|^{\theta}.$$

- $f$ is twice differentiable with $f''$ Lipschitz on $(x-\varepsilon_0, x+\varepsilon_0)$.
- Let $(Z_k)_{k\geqslant 1}$ be a sequence of i.i.d. random vectors with the same distribution as $Z$.
- Then, for every $\varepsilon \in (0,\varepsilon_0)$, the *RMSE* satisfies

$$\left\| f'(x) - \frac{1}{M}\sum_{k=1}^{M}\frac{F(x+\varepsilon,Z_k)-F(x-\varepsilon,Z_k)}{2\varepsilon}\right\|_2 \leqslant \sqrt{\left([f'']_{\rm Lip}\frac{\varepsilon^2}{2}\right)^2 + \frac{C^2_{Hol,F,Z}}{(2\varepsilon)^{2(1-\theta)}M}}$$

$$\leqslant [f'']_{\rm Lip}\frac{\varepsilon^2}{2} + \frac{C_{Hol,F,Z}}{(2\varepsilon)^{1-\theta}\sqrt{M}}.$$

# ℵ Practitioner's corner

- To divide the error by a 2-factor

$$\varepsilon \rightsquigarrow \varepsilon/\sqrt{2} \quad \text{and} \quad M \rightsquigarrow 2^{3-\theta} M = 2^{1-\theta} \cdot 4M$$

- **Exercise.** Prove the above Proposition.