RR with consistent increments *versus* crude *MC*

- Extreme at-the-money Call in a *B-S* model with

$$r = 0.15, \quad \sigma = 1.0, \quad T = 1, \quad X_0 = K = 100.$$

- Reference price $= C_0^{BS} = 42.9571$.
- Euler scheme with step size $h = \frac{T}{n}$, $n = 8$.
- Same complexity $\Rightarrow$ *B-S* Euler scheme $\rightsquigarrow 3M$ samples *versus* *RR*-Extrapolation meta-scheme $\rightsquigarrow M$ consistent samples.

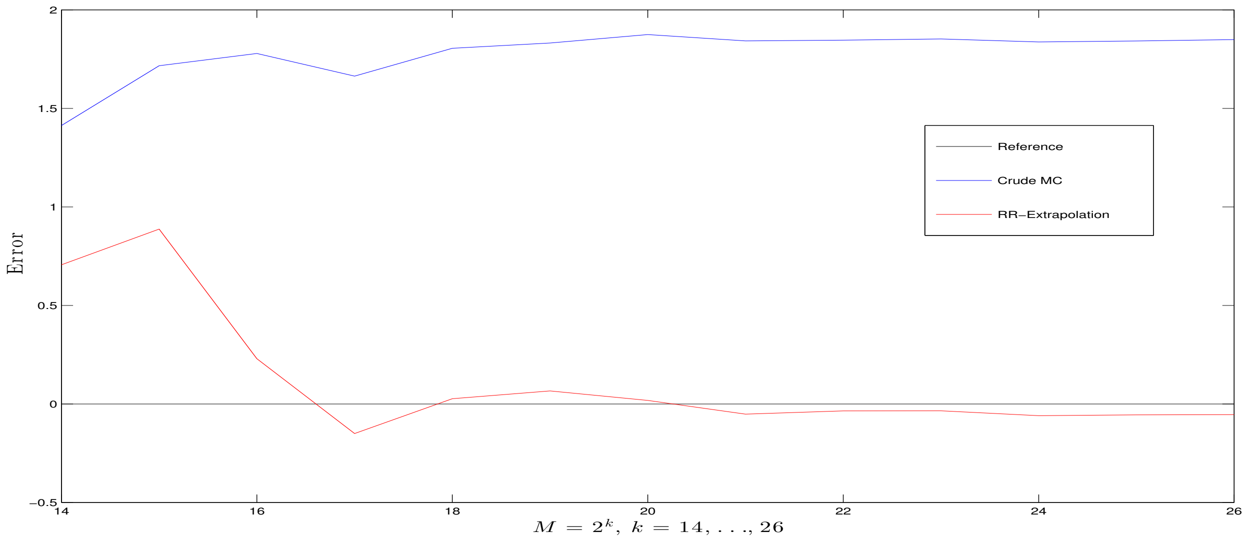

Figure: Richardson-Romberg extrapolation with consistent Brownian increments: solid blue = Crude MC on Euler scheme ($3M$); solid red= RR-Extrapol (M).

RR with non-consistent increments *vs* crude *MC*

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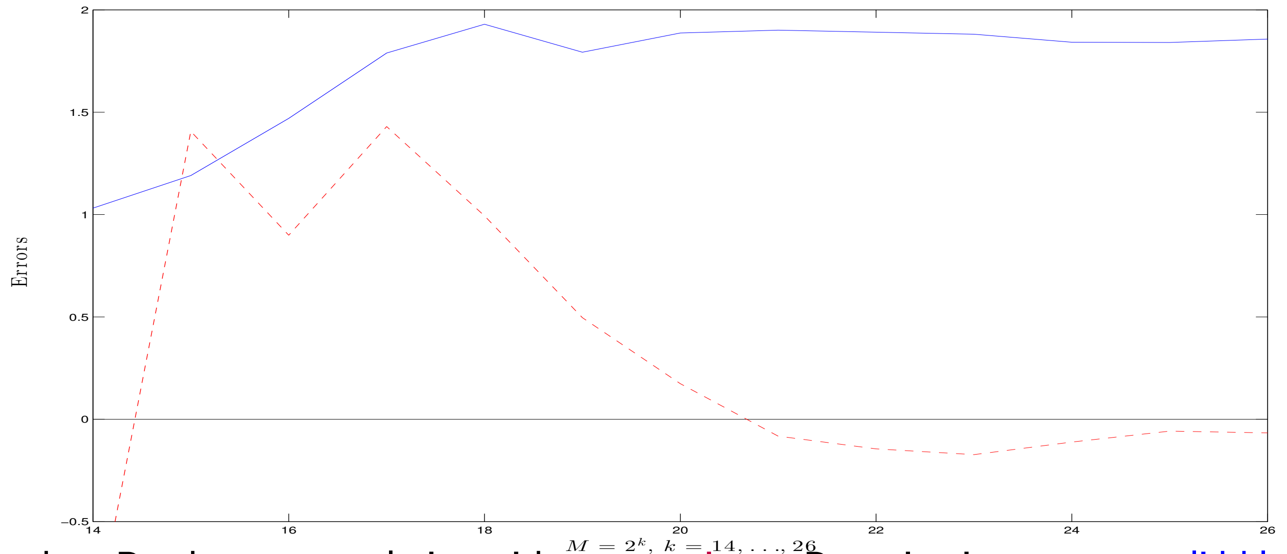

Figure: Richardson-Romberg extrapolation with non-consistent Brownian increments: solid blue = Crude MC ($3M$); dotted red = RR-Extrapol (M)

The case of path-dependent payoffs

- No similar general weak error result holds path-dependent functionals $F(X)$ where

$$F : \mathbb{D}([0,T],\mathbb{R}^d) = \big\{x : [0,T] \to \mathbb{R}^d,\ \text{càdlàg}\big\} \longrightarrow \mathbb{R}_+.$$

- Typical payoff functionals in Finance:
  - Asian payoffs: $F(x) = f\left(x(T), \dfrac{1}{T}\displaystyle\int_0^T x(t)dt\right)$.
  - Lookback payoffs: $F(x) = f\big(x(T), \sup_{t\in[0,T]} x(t)\big)$, $F(x) = f\big(x(T), \inf_{t\in[0,T]} x(t)\big)$.
  - Barrier payoffs:
    - ⋆ $F(x) = f\big(x(T)\big)\mathbf{1}_{\{\sup_{t\in[\mathbf{0},T]} x(t)\leqslant L\}}$ or $F(x) = f\big(x(T)\big)\mathbf{1}_{\{\inf_{t\in[\mathbf{0},T]} x(t)\geqslant L\}}$ are knock-out-payoffs
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    - ⋆ More generally $F(x) = f\big(x(T)\big)\mathbf{1}_{\{\tau_D > T\}}$

$$\text{with}\quad \tau_D(x) = \inf\big\{s : x(s) \notin D\big\},\quad D \subset \mathbb{R}^d,\ D \text{ open set.}$$

- Keep in mind that the problem is to "expand" for the Euler schemes of a Brownian diffusion process $X = (X_t)_{t\in[0,T]}$

$$\mathbb{E}\, F\big((\bar{X}_t)_{t\in[0,T]}\big) - \mathbb{E}\, F\big((X_t)_{t\in[0,T]}\big) = ? \quad \text{as} \quad h = \frac{T}{n} \to 0,$$

or

$$\mathbb{E}\, F\big((\widetilde{X}_t)_{t\in[0,T]}\big) - \mathbb{E}\, F\big((X_t)_{t\in[0,T]}\big) = ? \quad \text{as} \quad h = \frac{T}{n} \to 0.$$

- This question has been investigated by several authors (e.g. E. Gobet for barrier like options) with (very) partial weak error results at order 1 (see detailed written notes), typically of the form:
  - Genuine Euler scheme (a priori not simulable/computable):

$$\mathbb{E}\, F\big((\bar{X}_t)_{t\in[0,T]}\big) = \mathbb{E}\, F\big((X_t)_{t\in[0,T]}\big) + c_1 h + o\big(h\big)$$

  - Stepwise constant Euler scheme (a priori simulable/computable):

$$\mathbb{E}\, F\big((\widetilde{X}_t)_{t\in[0,T]}\big) = \mathbb{E}\, F\big((X_t)_{t\in[0,T]}\big) + c_1 h^{\frac{1}{2}} + o\big(h^{\frac{1}{2}}\big).$$

Theorem (Example of barrier options)

(a) If the domain D is bounded and has a smooth enough boundary (in fact $\mathcal{C}^3$), if $b \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathcal{C}^3(\mathbb{R}^d, \mathcal{M}(d,q,\mathbb{R}))$, σ uniformly elliptic on D (i.e. $\sigma\sigma^*(x) \geqslant \varepsilon_0 I_d$, $\varepsilon_0 > 0$), then, for every bounded Borel function f vanishing in a neighborhood of ∂D,

$$\mathbb{E}\Big(f(X_T)\mathbf{1}_{\{\tau_D(X)>T\}}\Big) - \mathbb{E}\Big(f(\widetilde{X}^n_T)\mathbf{1}_{\{\tau_D(\widetilde{X}^n)>T\}}\Big) = O\left(\frac{1}{\sqrt{n}}\right) \text{ as } n \to +\infty$$

where $\widetilde{X}$ denotes the stepwise constant Euler scheme.

(b) If, furthermore, b and σ are $\mathcal{C}^5$ and D is a half-space, then the genuine Euler scheme *$\bar{X}^n$ satisfies*

$$\mathbb{E}\Big(f(X_T)\mathbf{1}_{\{\tau_D(X)>T\}}\Big) - \mathbb{E}\Big(f(\bar{X}^n_T)\mathbf{1}_{\{\tau_D(\bar{X}^n)>T\}}\Big) = O\left(\frac{1}{n}\right) \text{ as } n \to +\infty.$$

- Unfortunately these assumptions are not satisfied by standard barrier options.

Lookback style options (conjectures and results)

- It is "suggested" in Seumen-Tonou's PhD (with a proof when $X = W$, otherwise a conjecture) that if b, $\sigma \in \mathcal{C}_b^4(\mathbb{R}, \mathbb{R})$, σ is uniformly elliptic and $f \in \mathcal{C}_{pol}^{4,2}(\mathbb{R}^2)$ (existing partial derivatives with polynomial growth):

$$\mathbb{E}\Big(f(X_T, \max_{t\in[0,T]} X_t)\Big) - \mathbb{E}\Big(f(\widetilde{X}^n_T, \max_{0\leqslant k\leqslant n} \widetilde{X}^n_{t_k})\Big) = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{as} \quad n \to +\infty.$$

- A similar improvement – $O(\frac{1}{n})$ rate – as above can be expected (but also remains a conjecture) when replacing $\widetilde{X}$ by the continuous Euler scheme $\bar{X}^n$, namely

$$\mathbb{E}\Big(f(X_T, \max_{t\in[0,T]} X_t)\Big) - \mathbb{E}\Big(f(\bar{X}^n_T, \max_{t\in[0,T]} \bar{X}^n_t)\Big) = O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to +\infty.$$

- More recently, Alfonsi et al. obtained $O(n^{-(\frac{2}{3}-\eta)})$ for every $\eta > 0$.
- Relies on transport inequalities and the Wasserstein metric.

- Nevertheless, is it possible to simulate some specific functionals of the genuine Euler scheme?
- Yes... when $F(x) = f\big(x(T), \sup_{t\in[0,T]} x(t)\big)$
- But it is a long way from home...

The Brownian bridge

Theorem (Brownian bridge)

Let $W = (W_t)_{t \geqslant 0}$ be a standard Brownian motion.

(a) *Let $T > 0$. Then, the so-called* standard Brownian bridge on $[0, T]$ *defined by*

$$Y_t^{W,T} := W_t - \frac{t}{T} W_T, \quad t \in [0, T], \tag{1}$$

is an $\mathcal{F}_T^W$-measurable centered Gaussian process, independent of $(W_{T+s})_{s \geqslant 0}$, whose distribution is characterized by its covariance structure

$$\mathbb{E}\left(Y_s^{W,T} Y_t^{W,T}\right) = s \wedge t - \frac{st}{T} = \frac{(s \wedge t)(T - s \vee t)}{T}, \qquad 0 \leqslant s,\ t \leqslant T.$$

To be continued (...)

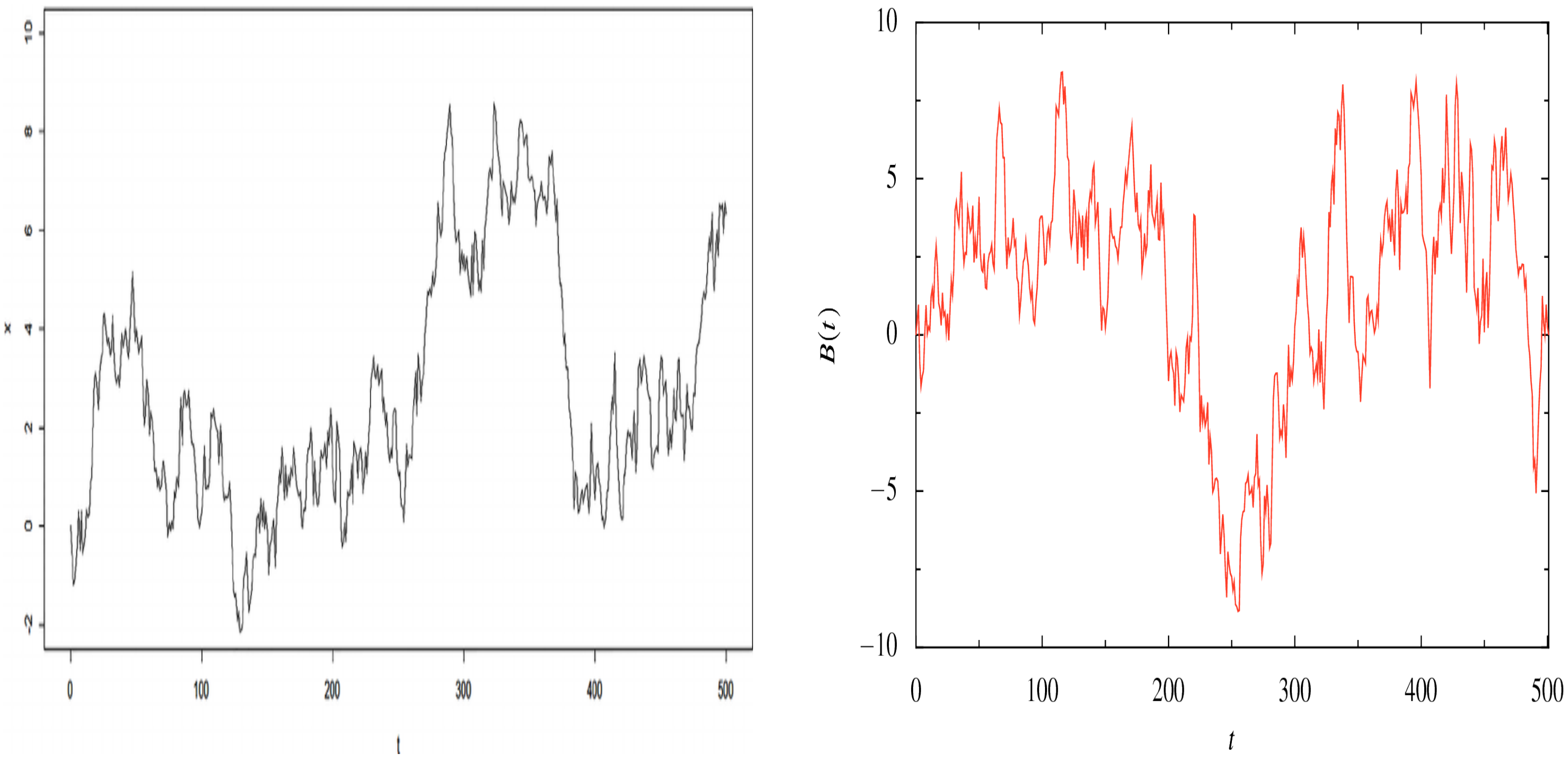

Figure: $T = 300$. Left : Standard Brownian motion path (black). Right: Standard Brownian bridge paths (red).

Proof

- $(Y_t^{W,T})_{t\in[0,T]}$ is clearly centered since W is. Starting from $\mathbb{E}\, W_s W_t = s\wedge t,\ s,\ t\in[0,T]\dots$
- Elementary computations show (Exercise!) that

$$\begin{aligned}\mathbb{E}\,(Y_t^{W,T}Y_s^{W,T}) &= \mathbb{E}\,W_tW_s - \frac{s}{T}\mathbb{E}\,W_tW_{\scriptscriptstyle T} - \frac{t}{T}\mathbb{E}\,W_sW_{\scriptscriptstyle T} + \frac{st}{T^2}\mathbb{E}W_{\scriptscriptstyle T}^2\\ &= s\wedge t - \frac{st}{T} - \frac{ts}{T} + \frac{ts}{T}\\ &= s\wedge t - \frac{ts}{T} = \frac{(s\wedge t)(T-s\vee t)}{T}.\end{aligned}$$

- Let $\mathbb{G}_W = \overline{\mathrm{span}\big\{W_t,\ t\geqslant 0\big\}}^{L^2(\mathbb{P})}$ be the closed vector subspace of $L^2(\Omega,\mathcal{A},\mathbb{P})$ spanned by the Brownian motion W.
- As W is a centered Gaussian process: independence $\Longleftrightarrow$ absence of correlation in $\mathbb{G}_W$.
- $Y_t^{W,T}$ belongs to this space by construction.
- As, for every $u\geqslant T$, $\mathbb{E}\,(Y_t^{W,T}W_u)=0$, $Y_t^{W,T}\perp \mathrm{span}(W_u,\ u\geqslant T)$.
- Consequently, $Y^{W,T}$ is independent of $(W_{T+u})_{u\geqslant 0}$.

Theorem (Brownian bridge)

(b) Let $T_0,\ T_1 \in (0,+\infty)$, $T_0 < T_1$. Then

$$\mathcal{L}\left((W_t)_{t\in[T_0,T_1]}\,|\,W_s,\ s\notin(T_0,T_1)\right) = \mathcal{L}\Big((W_t)_{t\in[T_0,T_1]}\,|\,W_{T_0}, W_{T_1}\Big)$$

so that $(W_t)_{t\in[T_0,T_1]}$ and $(W_s)_{s\notin(T_0,T_1)}$ are independent given (W_{T_0}, W_{T_1}). Moreover, this conditional distribution is given by

$$\mathcal{L}\Big((W_t)_{t\in[T_0,T_1]}\,|\,W_{T_0}=x, W_{T_1}=y\Big) = \mathcal{L}\left(\underbrace{x+\frac{t-T_0}{T_1-T_0}(y-x)}_{\textit{interpolation}} + \underbrace{\big(Y^{B,(T_1-T_0)}_{t-T_0}\big)_{t\in[T_0,T_1]}}_{\textit{Brownian bridge}}\right),$$

where B is a generic standard Brownian motion.

- Pathwise $B_t = W_{T_0+t} - W_{T_0}$.

Proof

- Write
$$W_t = W_{T_0} + W^{(T_0)}_{t-T_0},$$
where $W^{(T_0)}_t = W_{T_0+t} - W_{T_0}$ standard B.M $\perp\!\!\!\perp \mathcal{F}^W_{T_0}$.
- Check that
$$W^{(T_0)}_s = \frac{s}{T_1 - T_0} W^{(T_0)}_{T_1-T_0} + Y^{W^{(T_0)},T_1-T_0}_s.$$
- Plugging this identity into the above equality at time $s = t - T_0$ leads to
$$W_t = W_{T_0} + \frac{t - T_0}{T_1 - T_0}(W_{T_1} - W_{T_0}) + Y^{W^{(T_0)},T_1-T_0}_{t-T_0}.$$
- $\widetilde{Y} := (Y^{W^{(T_0)},T_1-T_0}_{t-T_0})_{t\in[T_0,T_1]}$ is a centered Gaussian process, $\mathcal{F}^{W^{(T_0)}}_{T_1-T_0}$-measurable by (a), hence independent of $\mathcal{F}^W_{T_0}$ since $W^{(T_0)}$ is.

Proof

- Consequently, $\widetilde{Y} \perp\!\!\!\perp (W_t)_{t\in[0,T_0]}$. Furthermore, $\widetilde{Y} \perp\!\!\!\perp (W^{(T_0)}_{T_1-T_0+u})_{u\geqslant 0}$ by (a).
- Hence it is $L^2(\mathbb{P})$-orthogonal to $W_{T_1+u} - W_{T_0}$, $u \geqslant 0$, in $\mathbb{G}_W$.
- As it is also orthogonal to W_{T_0} in $\mathbb{G}_W$ since W_{T_0} is $\mathcal{F}^W_{T_0}$-measurable, it is orthogonal to $W_{T_1+u} = W^{(T_0)}_{T_1-T_0+u} + W_{T_0}$, $u \geqslant 0$.
- Which in turn implies independence of $\widetilde{Y}$ and $(W_{T_1+u})_{u\geqslant 0}$ since all these random variables lie in $\mathbb{G}_W$.
- Finally, the same argument – in $\mathbb{G}_W$, no correlation implies independence – implies that $\widetilde{Y}$ is independent of $(W_s)_{s\in\mathbb{R}_+\setminus(T_0,T_1)}$ *i.e.* of the σ-field $\sigma(W_s,\, s \notin (T_0, T_1))$.
- The end of the proof follows from the above identity.

Bridge of the Euler scheme (diffusion bridge)

Let $\bar{X} = (\bar{X}^n_t)_{t\in[0,T]}$ be the genuine Euler scheme and $t^n_k = \frac{kT}{n}$, $k = 0 : n$.

Theorem (Euler scheme bridge)

Assume that $\sigma(t,x) \neq 0$ for every $t \in [0,T]$, $x \in \mathbb{R}$.

(a) The processes $(\bar{X}^n_t)_{t\in[t^n_k, t^n_{k+1}]}$, $k = 0, \ldots, n-1$, are conditionally independent given the discrete time Euler scheme $\bar{X}^n_{t^n_k}$, $k = 0, \ldots, n$.

(b) Furthermore, for every $k \in \{0, \ldots, n\}$, the conditional distribution

$$\begin{aligned}
&\mathcal{L}\Big((\bar{X}^n_t)_{t\in[t^n_k, t^n_{k+1}]} \,\big|\, \bar{X}^n_{t^n_\ell} = x_\ell,\ \ell = 0, \ldots, n\Big)\\
&\qquad = \mathcal{L}\Big((\bar{X}^n_t)_{t\in[t^n_k, t^n_{k+1}]} \,\big|\, \bar{X}^n_{t^n_k} = x_k, \bar{X}_{t^n_{k+1}} = x_{k+1}\Big)\\
&\qquad = \mathcal{L}\left(\left(x_k + \frac{n(t-t^n_k)}{T}(x_{k+1} - x_k) + \sigma(t^n_k, x_k) Y^{B,T/n}_{t-t^n_k}\right)_{t\in[t^n_k, t^n_{k+1}]}\right)
\end{aligned}$$

where $(Y^{B,T/n}_s)_{s\in[0,T/n]}$ is a Brownian bridge.

(A bit) More on the Diffusion bridge

The distribution of this Gaussian process (sometimes called a diffusion bridge) is entirely characterized by:

- its expectation function $\left(x_k + \frac{n(t-t_k^n)}{T}(x_{k+1}-x_k)\right)_{t\in[t_k^n,t_{k+1}^n]}$,
- its covariance operator $\sigma^2(t_k^n, x_k)\dfrac{(s\wedge t - t_k^n)(t_{k+1}^n - s\vee t)}{t_{k+1}^n - t_k^n}$,

  $s, t \in [t_k^n, t_{k+1}^n]$.

Supremum of a diffusion bridge

Proposition

The distribution of the supremum of the Brownian bridge starting at 0 and arriving at y at time T, defined by $Y_t^{W,T,y} = \frac{t}{T}y + W_t - \frac{t}{T}W_T$ on $[0, T]$, is given by

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Y_t^{W,T,y} \leqslant z\Big) = \begin{cases} 1 - \exp\left(-\frac{2}{T}z(z-y)\right) & \text{if } z \geqslant \max(y,0), \\ 0 & \text{if } z \leqslant \max(y,0). \end{cases}$$

The key of the proof is the reflection principle.