Numerical Probability

Monte Carlo simulation II (biased)

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Foreword

- So far, the courses were devoted to variance and how to reduce it ?
- Today is devoted to the bias and how to kill it ?
- Can we do both ?

Brownian diffusion

- Stochastic Differential Equation (*SDE*)

$$(SDE) \quad \equiv \quad dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where

- $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R})$ are continuous,
- $W = (W_t)_{t\in[0,T]}$ denotes a q-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.,
- $X_0 : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ is a random vector, *independent of* W,
- b and σ are Lipschitz continuous in x uniformly in $t \in [0, T]$, *i.e.*

$$\forall\, x, y \in \mathbb{R}^d, \quad \big|b(t, x) - b(t, y)\big| + \big\|\sigma(t, x) - \sigma(t, y)\big\| \leqslant K|x - y|.$$

Remark. $|\,.\,|$ canonical Euclidean norm on $\mathbb{R}^d$ and $\|\,.\,\|$ denote the canonical Fröbenius norm on $\mathcal{M}(d, q, \mathbb{R})$ i.e. $\|A\| = \sqrt{\mathrm{Tr}(AA^*)}$.

Filtration

- If $X_0 = x_0$ is deterministic, we consider

$$\mathcal{F}_t^W = \sigma\big(\mathcal{N}_{\mathbb{P}},\ W_s,\ 0 \leqslant s \leqslant t\big),\ \ t \in [0, T]$$

  where $\mathcal{N}_{\mathbb{P}}$ denotes the class of $\mathbb{P}$-negligible sets of $\mathcal{A}$ (*i.e.* all negligible sets if the σ-algebra $\mathcal{A}$ is supposed to be $\mathbb{P}$-complete).

- Owing to Kolmogorov's 0-1 law, $(\mathcal{F}_t^W)$ is right continuous, *i.e.* $\mathcal{F}_t^W = \cap_{s>t}\mathcal{F}_s^W$ for every $t \in [0, T)$ (satisfies "usual conditions").

- In the general case, we consider the augmented filtration

$$\mathcal{F}_t := \sigma\big(X_0, \mathcal{N}_{\mathbb{P}},\ W_s,\ 0 \leqslant s \leqslant t\big),\ \ t \in [0, T],$$

  which also satisfies "usual conditions".

Theorem (Strong solution of *SDE*)

Under the above assumptions on b, σ, X_0 and W, the above SDE has a unique $(\mathcal{F}_t)$-adapted solution $X = (X_t)_{t\in[0,T]}$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, starting from X_0 at time 0, in the following sense:

$$\mathbb{P}\text{-a.s.} \quad \forall\, t \in [0, T], \quad X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

This solution has $\mathbb{P}$-a.s. continuous paths (in fact α-Hölder paths, $\alpha \in (0, \frac{1}{2})$).

Back to the assumptions

Lipschitz continuity can be relaxed into

- Local Lipschitz continuity: For every $N \in \mathbb{N}^*$, $\exists\, K_N \in (0, +\infty)$,

$$\forall\, t \in [0, T],\ \forall\, x,\ y \in \mathbb{R}^d, |x|,\ |y| \leqslant N \Longrightarrow$$

$$\big|b(t,x) - b(t,y)\big| + \big\|\sigma(t,x) - \sigma(t,y)\big\| \leqslant K_N |x-y|$$

- Linear growth:

$$\forall\, t \in [0, T],\ \forall\, x \in \mathbb{R}^d, \quad \big|b(t,x)\big| + \big\|\sigma(t,x)\big\| \leqslant C(1+|x|).$$

But this is not the right framework for discretization schemes...

Discrete time Euler scheme

- Like for *ODE*s in the deterministic framework, we *freeze the* coefficients of the *SDE* between regularly spaced discretization instants $t_k = t_k^n = \frac{kT}{n}$, $k = 0 : n$.
- The Euler is defined by induction: for $k = 0 : n-1$,

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \frac{T}{n} b(t_k^n, \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n})\big(W_{t_{k+1}^n} - W_{t_k^n}\big), \ \bar{X}_0 = X_0.$$

- Practitioner's viewpoint: for $k = 0 : n-1$,

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \frac{T}{n} b(t_k^n, \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n})\sqrt{\frac{T}{n}}\, Z_{k+1}, \ \bar{X}_0 = X_0.$$

  where $(Z_k)_{1\leqslant k\leqslant n}$ is i.i.d. $\mathcal{N}(0; I_q)$-distributed...
- The sequence $(Z_k)_{1\leqslant k\leqslant n}$ is given by

$$Z_k = Z_k^n := \sqrt{\frac{n}{T}}\big(W_{t_k^n} - W_{t_{k-1}^n}\big), \quad k = 1, \ldots, n.$$

Stepwise constant Euler scheme

- To alleviate notations, we denote from now on

$$\underline{t} := t_k^n = \frac{kT}{n}, \qquad \text{if } t \in [t_k^n, t_{k+1}^n).$$

- The *stepwise constant Euler scheme*, denoted $(\widetilde{X}_t)_{t\in[0,T]}$ for convenience, is defined by

$$\widetilde{X}_t = \bar{X}_{\underline{t}}, \quad t \in [0, T].$$

Genuine (or continuous) Euler scheme

- How to extend the Euler scheme into a continuous process over $[0, T]$?
- Idea: interpolating the drift and the diffusion term in their own scale
- Namely, set for every $k \in \{0, \ldots, n-1\}$ and every $t \in [t_k^n, t_{k+1}^n)$,

$$\bar{X}_t = \bar{X}_{\underline{t}} + (t - \underline{t})b(\underline{t}, \bar{X}_{\underline{t}}) + \sigma(\underline{t}, \bar{X}_{\underline{t}})(W_t - W_{\underline{t}}), \quad \bar{X}_0 = X_0.$$

- It is clear that $\lim\limits_{t \to t_{k+1}^n, t < t_{k+1}^n} \bar{X}_t = \bar{X}_{t_{k+1}^n}$ since W has continuous paths.
- Thus $(\bar{X}_t)_{t \in [0,T]}$ is an $\mathcal{F}_t$-adapted process with continuous paths.

Genuine Euler scheme as an Itô Process

If b and σ are continuous functions on $[0, T] \times \mathbb{R}^d$. Then,

$$\bar{X}_t = X_0 + \int_0^t b(\underline{s}, \bar{X}_{\underline{s}})ds + \int_0^t \sigma(\underline{s}, \bar{X}_{\underline{s}})dW_s, \; t \in [0, T].$$

- The naive simulation of the *paths of the* genuine Euler scheme is not possible.

- It is mainly a theoretical tool.

- However, some functionals of the genuine Euler may sometimes be simulable like

$$\Big(\bar{X}_{t_k^n}, k = 0:n, \sup_{t\in[0,T]} \bar{X}_t\Big) \quad \text{or} \quad \Big(\bar{X}_{t_k^n}, k = 0:n, \inf_{t\in[0,T]} \bar{X}_t\Big)$$

using the diffusion bridge method.

- But in general to simulate a functional, one has to switch to

$$F\big((\widetilde{X}_t^n)_{t\in[0,T]}\big)\dots$$

provided $F : \mathbb{D}([0,T],\mathbb{R}^d) \to \mathbb{R}$ can be computed on càdlàg functions.

Moment control

Theorem (Polynomial moments)

Assume that the coefficients b and σ of the SDE are Borel functions simply satisfying the linear growth assumption:

$$\forall\, t \in [0, T],\ \forall\, x \in \mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \leqslant C(1+|x|)$$

for some real constant $C = C_T > 0$ and a "horizon" $T > 0$.

For every $p \in (0, +\infty)$, there exists $\kappa_p > 0$ s.t. every strong solution $(X_t)_{t\in[0,T]}$ of the SDE (if any) satisfies

$$\Big\| \sup_{t\in[0,T]} |X_t| \Big\|_p \leqslant 2\, e^{\kappa_p CT}(1 + \|X_0\|_p)$$

$$\text{and } \forall\, n \geqslant 1, \quad \Big\| \max_{0\leqslant k\leqslant n} |\bar{X}^n_{t^n_k}| \Big\|_p \leqslant \Big\| \sup_{t\in[0,T]} |\bar{X}^n_t| \Big\|_p \leqslant 2\, e^{\kappa_p CT}(1 + \|X_0\|_p).$$

Convergence (genuine Euler)

- We make/need the more stringent assumption

$$(H_T^\beta) \equiv \begin{cases} \exists\, C_{b,\sigma,T} > 0 \ \text{ such that } \forall s,t \in [0,T],\ \forall\, x,y \in \mathbb{R}^d, \\ (i)\ |b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq C_{b,\sigma,T}|y-x|, \\ (ii)\ |b(t,x) - b(s,x)| + \|\sigma(t,x) - \sigma(s,x)\| \leq C_{b,\sigma,T}\big(1+|x|\big)|t-s|^\beta. \end{cases}$$

When $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$ $(H_T^\beta) \equiv (H_T^1)$.

- Then,

Theorem (Strong convergence rate for the genuine Euler scheme)

(a) Assume b and σ satisfy (H_T^β) for some $\beta \in (0,1]$. Then, there exists a universal function $K = K(p,b,\sigma,T)$ such that, for every $n \geqslant T$,

$$\Big\| \sup_{t\in[0,T]} |X_t - \bar{X}_t^n| \Big\|_p \leq K(p,b,\sigma,T)\big(1 + \|X_0\|_p\big)\left(\frac{T}{n}\right)^{\beta\wedge\frac{1}{2}}.$$

Strong Rate for the discrete time Euler scheme

Theorem

(*b*) *In particular the former error bound is satisfied when the supremum is restricted to discretization instants* t_k^n*, namely*

$$\left\|X_T - \bar{X}_T^n\right\|_p \leqslant \left\|\max_{0\leqslant k\leqslant n} |X_{t_k} - \bar{X}_{t_k}^n|\right\|_p \leq K(p,b,\sigma,T)(1+\|X_0\|_p)\left(\frac{T}{n}\right)^{\beta\wedge\frac{1}{2}}$$

with the same constant $K(p,b,\sigma,T)$ *(independent of* n*).*

- Satisfactory for marginal (or *vanilla*) simulation of type $f(\bar{X}_T^n)$ corresponding to $t_n^n = T$.
- Not satisfactory for *path-dependent functionals*.
- What happens with the stepwise constant Euler scheme?

Stepwise constant Euler scheme I

First we compare both stepwise constant and genuine Euler schemes.

Theorem (Stepwise versus genuine Euler scheme)

(a) *If b and σ satisfy the linear growth assumption with a real constant $L_{b,\sigma,T} > 0$, then, for every $p \in (0, +\infty)$ and every $n \geqslant T$,*

$$\Big\| \sup_{t\in[0,T]} \big|\bar{X}^n_t - \underbrace{\bar{X}^n_{\underline{t}}}_{=\widetilde{X}^n_t}\big| \Big\|_p \le \tilde{\kappa}_p e^{\tilde{\kappa}_p L_{b,\sigma,T} T}\big(1 + \|X_0\|_p\big)\sqrt{\frac{T}{n}}\sqrt{(1+\log\, n)}.$$

Theorem (Stepwise versus diffusion)

(*b*) *In particular if b and σ satisfy Assumptions (H_T^β), for every $n \geqslant T$,*

$$\Big\|\sup_{t\in[0,T]}|X_t-\widetilde{X}_t^n|\Big\|_p \leqslant \widetilde{K}(p,b,\sigma,T)\big(1+\|X_0\|_p\big)\left[\sqrt{\frac{T(1+\log n)}{n}}+\left(\frac{T}{n}\right)^{\beta\wedge\frac{1}{2}}\right]$$
$$=O\left(\left(\frac{1}{n}\right)^{\beta}+\sqrt{\frac{1+\log n}{n}}\right)$$

where

$$\widetilde{K}(p,b,\sigma,T)=\tilde{\kappa}'_p(1+C'_{b,\sigma,T})^2 e^{\tilde{\kappa}'_p(1+C'_{b,\sigma,T})T},$$

$\tilde{\kappa}'_p > 0$ *is a positive universal real constant only depending on p (increasing in p) and $C'_{b,\sigma,T}$ only depends on $C_{b,\sigma,T}$ and $\|b(.,0)\|_{\sup}$, $\|\sigma(.,0)\|_{\sup}$.*

Stepwise constant Euler scheme II: Optimality ?

- Imagine the simplest case : $d = q = 1$ and $b \equiv 0$ and $\sigma = 1$ *i.e.* $X = W$!
- Then

$$\bar{W}^n_t - \widetilde{W}^n_t = W_t - W_{\underline{t}},\ t \in [0, T]$$

and

$$\Big\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \Big\|_p = \Big\| \max_{k=1,\dots,n} \sup_{t\in[t^n_{k-1},t^n_k)} |W_t - W_{t^n_{k-1}}| \Big\|_p$$

$$= \sqrt{\frac{T}{n}} \Big\| \max_{k=1,\dots,n} \sup_{t\in[k-1,k)} |B_t - B_{k-1}| \Big\|_p$$

where $B_t := \sqrt{\frac{n}{T}} W_{\frac{T}{n}t}$ is a standard B.M. (scaling property). Hence

$$\Big\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \Big\|_p = \sqrt{\frac{T}{n}} \Big\| \max_{k=1,\dots,n} \zeta_k \Big\|_p \dots$$

- …where $\zeta_k := \sup_{t\in[k-1,k)} |B_t - B_{k-1}|$ are i.i.d.
- Now
$$\zeta_k \geqslant Z_k := |B_k - B_{k-1}| \stackrel{d}{=} |B_1|.$$
- Now, for every $p \geqslant 2$,
$$\Big\| \max_{k=1,\ldots,n} |Z_k| \Big\|_p = \sqrt{\big\| \max_{k=1,\ldots,n} Z_k^2 \big\|_{\frac{p}{2}}}.$$

The (Z_k^2) are i.i.d. with $Z_1^2 \stackrel{d}{=} B_1^2 \stackrel{d}{=} \chi^2(1) = \dfrac{e^{-\frac{u}{2}}}{\sqrt{2\pi u}} \mathbf{1}_{\{u>0\}} du$.

- An integration by parts yields the *survival* distribution function
$$\bar F_{\chi^2(1)}(z) = \mathbb{P}(\chi^2(1) > z) = \int_z^{+\infty} \frac{e^{-\frac{u}{2}}}{\sqrt{2\pi u}} du, \ z \geqslant 0$$

satisfies
$$\bar F_{\chi^2(1)}(z) = \frac{2e^{-\frac{z}{2}}}{\sqrt{2\pi z}} - \int_z^{+\infty} \frac{e^{-\frac{u}{2}}}{u\sqrt{2\pi u}} du \sim \frac{2e^{-\frac{z}{2}}}{\sqrt{2\pi z}} \quad \text{as} \quad z \to +\infty.$$

- so that, there exists $a > 0$,

$$\forall z \geqslant a > 0, \quad \overline{F}_{\chi^2(1)}(z) = \mathbb{P}(\chi^2(1) > z) \geqslant c\, f_{\chi^2(1)}(z).$$

- For any such distribution, if $F = F_{\chi^2(1)}$ denotes the cumulative distribution function of Z_1^2,

$$\begin{aligned}
\Big\| \max(Z_1^2, \ldots, Z_n^2) \Big\|_1 &= \int_0^{+\infty} \mathbb{P}\big(\max_{1 \leqslant k \leqslant n} Z_k^2 \geqslant u\big) du \\
&= \int_0^{+\infty} \Big(1 - \mathbb{P}\big(\max_{1 \leqslant k \leqslant n} Z_k^2 \leqslant u\big)\Big) du \\
&= \int_0^{+\infty} \big(1 - F^n(u)\big) du \\
&\geqslant \int_a^{+\infty} \big(1 - F^n(u)\big) du \\
&= \sum_{k=1}^{n} \int_a^{+\infty} F^{k-1}(u)\big(1 - F(u)\big) du \\
&= \sum_{k=1}^{n} \int_a^{+\infty} F^{k-1}(u) \overline{F}(u) du.
\end{aligned}$$

- Hence, using that $\overline{F}(u) = \overline{F}_{\chi^2(1)}(u) \geqslant c\, f_{\chi^2(1)}(u)$ yields

$$\left\| \max(Z_1^2, \ldots, Z_n^2) \right\|_1 \geqslant c \sum_{k=1}^{n} \int_a^{+\infty} F^{k-1}(u) f_{\chi^2(1)}(u)du$$
$$= c \sum_{k=1}^{n} \frac{1^k - F^k(a)}{k}$$
$$\geqslant c \Big(\log(n+1) + \log\big(1 - F(a)\big) \Big).$$

- **Conclusion.** Finally, at least for every $p \geqslant 2$,

$$\Big\| \sup_{t \in [0,T]} |\bar{W}_t^n - \widetilde{W}_t^n| \Big\|_p \geqslant c_p \sqrt{\frac{T}{n}(1 + \log n)}.$$

The rate of convergence of the stepwise constant Euler scheme $\widetilde{X}^n$ toward X in L^p cannot be (universally) improved.

Upper-bound

Lemma

Let $Y_1, \ldots Y_n$ be non-negative random variables with the same distribution satisfying $\mathbb{E}\,(e^{\lambda Y_1}) < +\infty$ for some $\lambda > 0$. Then,

$$\forall\, r \in (0, +\infty), \qquad \big\|\max(Y_1, \ldots Y_n)\big\|_r \leq \frac{1}{\lambda}\big(\log n + C_{r,Y_1,\lambda}\big).$$

- Let us come back to the i.i.d. sequence $\zeta_k = \sup_{[k-1,k)} |B_t - B_{k-1}|$.
- We know that $B \stackrel{d}{=} -B$ so that

$$e^{\lambda\zeta_1^2} \leqslant e^{\lambda(B_1^*)^2} + e^{\lambda((-B)_1^*)^2}$$

where $B_t^* := \sup\limits_{0\leqslant s\leqslant t} B_s \stackrel{d}{=} |B_t|$ (true but not trivial, cf. Revuz-Yor's book or Shi's Course).

- Now, for every $\lambda \in [0, 1/2)$,

$$\mathbb{E}\, e^{\lambda(B_1^*)^2} = \mathbb{E}\, e^{\lambda|B_1|^2} = \int e^{\lambda z^2 - \frac{1}{2}z^2}\frac{dz}{\sqrt{2\pi}} = \frac{1}{\sqrt{1-2\lambda}} < +\infty.$$

- Then, for every $\lambda \in [0, 1/2)$,

$$\mathbb{E}\, e^{\lambda \zeta_1^2} \leqslant 2\mathbb{E}\, e^{\lambda (B_1^*)^2} < +\infty.$$

- We come back that to

$$\Big\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \Big\|_p = \sqrt{\frac{T}{n}} \Big\| \max_{k=1,\dots,n} \zeta_k \Big\|_p$$
$$= \sqrt{\frac{T}{n}} \sqrt{\Big\| \max_{k=1,\dots,n} \zeta_k^2 \Big\|_{\frac{p}{2}}}$$

- One concludes using the lemma with $r = \frac{p}{2}$.

Proof of the lemma for $r = 1$ (i.e. $p = 2$)

- One has, for any $\lambda > 0$,

$$\begin{aligned}\max(Y_1, \ldots, Y_n) &= \frac{1}{\lambda}\max(\lambda Y_1, \ldots, \lambda Y_n)\\ &= \frac{1}{\lambda}\log e^{\max(\lambda Y_1,\ldots,\lambda Y_n)}\end{aligned}$$

so that (owing to Jensen's Inequality in the 2^{nd} line)

$$\begin{aligned}\mathbb{E}\max(Y_1, \ldots, Y_n) &\leqslant \frac{1}{\lambda}\mathbb{E}\log\left(e^{\max(\lambda Y_1,\ldots,\lambda Y_n)}\right)\\ &\leqslant \frac{1}{\lambda}\log\left(\mathbb{E}\,e^{\max(\lambda Y_1,\ldots,\lambda Y_n)}\right)\\ &\leqslant \frac{1}{\lambda}\log\left(\mathbb{E}\,e^{\lambda Y_1} + \cdots \mathbb{E}\,e^{\lambda Y_n}\right)\\ &= \frac{1}{\lambda}\log\left(n\mathbb{E}\,e^{\lambda Y_1}\right)\\ &= \frac{\log n + \log(\mathbb{E}\,e^{\lambda Y_1}\ )}{\lambda}.\end{aligned}$$

Proof of the moment control theorem (preliminaries I)

- Here we provide a simpler proof when $d = q = 1$ and $p = 2$ and

$$b(t,x) = b(x) \quad \text{and} \quad \sigma(t,x) = \sigma(x).$$

- General case: see *Numerical Probability* book or notebook.

Lemma (Grönwall's Lemma)

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel non-negative locally bounded function and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying

$$(G) \quad \equiv \quad \forall\, t \geqslant 0, \quad f(t) \leqslant \alpha \int_0^t f(s)\, ds + \psi(t)$$

for a real constant $\alpha > 0$. Then

$$\forall\, t \geqslant 0, \qquad \sup_{0 \leqslant s \leqslant t} f(s) \leqslant e^{\alpha t}\psi(t).$$

Preliminaries II: Doob's inequality

- Now we recall the quadratic Doob Inequality which is needed in the proof (instead of the more sophisticated Burkhölder–Davis–Gundy required in the non-quadratic case).

Theorem (Doob's Inequality (L^2 case))

(a) Let $M = (M_t)_{t\geqslant 0}$ be a continuous martingale with $M_0 = 0$. Then, for every $T > 0$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} M_t^2\right) \leqslant 4\,\mathbb{E}\,M_T^2 = 4\,\mathbb{E}\,\langle M\rangle_T.$$

(b) If M is simply a continuous local *martingale with $M_0 = 0$, then, for every $T > 0$,*

$$\mathbb{E}\left(\sup_{t\in[0,T]} M_t^2\right) \leqslant 4\,\mathbb{E}\,\langle M\rangle_T.$$

Moment control (Bis!)

Theorem (Polynomial moments)

Assume that the coefficients b and σ of the SDE are Borel functions simply satisfying the linear growth assumption:

$$\forall\, t \in [0,T],\ \forall\, x \in \mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \leqslant C(1+|x|)$$

for some real constant $C = C_T > 0$ and a "horizon" $T > 0$.

For every $p \in (0,+\infty)$, there exists $\kappa_p > 0$ s.t. every strong solution $(X_t)_{t\in[0,T]}$ of the SDE (if any) satisfies

$$\Big\| \sup_{t\in[0,T]} |X_t| \Big\|_p \leqslant 2\, e^{\kappa_p C T}(1 + \|X_0\|_p)$$

$$\text{and } \forall\, n \geqslant 1, \quad \Big\| \max_{0\leqslant k\leqslant n} |\bar{X}^n_{t^n_k}| \Big\|_p \leqslant \Big\| \sup_{t\in[0,T]} |\bar{X}^n_t| \Big\|_p \leqslant 2\, e^{\kappa_p C T}(1 + \|X_0\|_p).$$

The proof (for the diffusion)

- We may assume w.l.g. $\mathbb{E}\, X_0^2 < +\infty$ (otherwise ... it is trivial).
- Let $\tau_L := \min\big\{t \,:\, |X_t - X_0| \geqslant L\big\}$, $L \in \mathbb{N}\setminus\{0\}$ (with $\min\varnothing = +\infty$).
- τ_L is an $(\mathcal{F}_t)_t$-stopping time (as the hitting time of a closed set by an $(\mathcal{F}_t)$-adapted process with continuous paths.
- For every $t \in [0, T]$,

$$|X_t^{\tau_L}| \leqslant L + |X_0|.$$

In particular, this implies that

$$(\star) \qquad \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{\tau_L}|^2\Big] \leqslant 2(L^2 + \mathbb{E}\, X_0^2) < +\infty.$$

$$\text{Then, } X_t^{\tau_L} = X_0 + \int_0^{t\wedge\tau_L} b(X_s)ds + \int_0^{t\wedge\tau_L} \sigma(X_s)dW_s$$

$$= X_0 + \int_0^{t\wedge\tau_L} b(X_s^{\tau_L})ds + \int_0^{t\wedge\tau_L} \sigma(X_s^{\tau_L})dW_s$$

owing to the local feature of both regular and stochastic integrals.

The proof II

- The stochastic integral

$$M_t^{(L)} := \int_0^{t\wedge\tau_L} \sigma(X_s^{\tau_L})dW_s$$

is a continuous local martingale null at zero with bracket process

$$\langle M^{(L)}\rangle_t = \int_0^{t\wedge\tau_L} \sigma^2(X_s^{\tau_L})ds.$$

- Now, using that $t\wedge\tau_L \leqslant t$, we derive that

$$|X_t^{\tau_L}| \leqslant |X_0| + \int_0^t |b(X_s^{\tau_L})|ds + \sup_{s\in[0,t]} \big|M_s^{(L)}\big|,$$

which in turn immediately implies that

$$\sup_{s\in[0,t]} |X_s^{\tau_L}| \leqslant |X_0| + \int_0^t |b(X_s^{\tau_L})|ds + \sup_{s\in[0,t]} |M_s^{(L)}|.$$

- The elementary inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, $a,\ b,\ c \geqslant 0$, combined with the Schwarz Inequality successively yields

$$\begin{aligned}
\sup_{s\in[0,t]} \left(X_s^{\tau_L}\right)^2 &\leqslant 3\left(X_0^2 + \left(\int_0^t |b(X_s^{\tau_L})|ds\right)^2 + \sup_{s\in[0,t]} |M_s^{(L)}|^2\right)\\
&\leqslant 3\left(X_0^2 + t\int_0^t |b(X_s^{\tau_L})|^2 ds + \sup_{s\in[0,t]} |M_s^{(L)}|^2\right).
\end{aligned}$$

- As Lipschitz continuous functions, b and σ have linear growth

$$|b(x)| + |\sigma(x)| \leqslant C_{b,\sigma}(1+|x|), \quad x\in\mathbb{R}.$$

- Taking expectation and applying Doob's Inequality to loc. mart. $M^{(L)}$

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} (X_s^{\tau_L})^2\right] &\leqslant 3\left(\mathbb{E}\, X_0^2 + TC_{b,\sigma}\int_0^t \left(1+\mathbb{E}\,|X_s^{\tau_L}|\right)^2 ds + 4\,\mathbb{E}\int_0^{t\wedge\tau_L} \sigma^2(X_s^{\tau_L})ds\right)\\
&\leqslant C_{b,\sigma,T}\left(\mathbb{E}\, X_0^2 + \int_0^t \left(1+\mathbb{E}\,|X_s^{\tau_L}|\right)ds + \mathbb{E}\int_0^t \left(1+|X_s^{\tau_L}|\right)^2 ds\right)\\
&= C_{b,\sigma,T}\left(\mathbb{E}\, X_0^2 + \int_0^t \left(1+\mathbb{E}\,|X_s^{\tau_L}|^2\right)ds\right).
\end{aligned}$$

where we used again (in the first two inequalities) that $\tau_L \wedge t \leqslant t$.

- Finally, this can be rewritten as

$$\mathbb{E}\Big[\sup_{s\in[0,t]}(X_s^{\tau_L})^2\Big] \leqslant C_{b,\sigma,T}\left(1+\mathbb{E}\,X_0^2+\int_0^t \mathbb{E}\left(|X_s^{\tau_L}|^2\right)ds\right)$$

- Then, Gronwall's Lemma applied to the bounded function (by $(\star)$!) $f_L(t) := \mathbb{E}\left(\sup_{s\in[0,t]}(X_s^{\tau_L})^2\right)$ at time $t=T$ implies

$$\forall\, L\geqslant 1,\quad \mathbb{E}\Big[\sup_{s\in[0,T]}(X_s^{\tau_L})^2\Big] \leqslant \underbrace{e^{C_{b,\sigma,T}\,T}C_{b,\sigma,T}}_{\text{free of } L\ !}\left(1+\mathbb{E}\,X_0^2\right).$$

- Now $\tau_L\uparrow+\infty$ *a.s.* as $L\uparrow+\infty$ since $\sup_{0\leqslant s\leqslant t}|X_s|<+\infty$ for every $t\geqslant 0$ *a.s.* owing to the continuity of the paths of X. Consequently,

$$\lim_{L\to+\infty}\sup_{s\in[0,T]}|X_s^{\tau_L}| = \sup_{s\in[0,T]}|X_s|.$$

- Fatou's Lemma implies

$$\mathbb{E}\Big[\sup_{s\in[0,T]}X_s^2\Big] \leqslant \liminf_{L\to+\infty}\mathbb{E}\Big[\sup_{s\in[0,T]}(X_s^{\tau_L})^2\Big] \leqslant C_{b,\sigma,T}\left(1+\mathbb{E}\,X_0^2\right)e^{C_{b,\sigma,T}\,T}$$

$$= C'_{b,\sigma,T}\left(1+\mathbb{E}\,X_0^2\right).$$

Proof III: Euler scheme

- As for the genuine Euler scheme (an Itô process), the proof follows closely the above lines, once we replace the stopping time τ_L by

$$\bar{\tau}_L = \min\left\{t \;:\; |\bar{X}_t - X_0| \geqslant L\right\}.$$

- It suffices to note that, for every $s \in [0,T]$, $\sup_{u\in[0,s]} |\bar{X}_{\underline{u}}| \leqslant \sup_{u\in[0,s]} |\bar{X}_u|$.

  Then one shows that

$$\sup_{n\geqslant 1} \mathbb{E}\Big[\sup_{s\in[0,T]} (\bar{X}^n_s)^2\Big] \leqslant C_{b,\sigma,T}\big(1+\mathbb{E}\,X_0^2\big)\,e^{C_{b,\sigma,T}\,T}.$$

- Details left as an EXERCISE.

L^2-Convergence rate: genuine Euler scheme

Theorem (Strong convergence rate for the genuine Euler scheme)

(a) *Assume b and σ satisfy (H_T^β) for some $\beta \in (0,1]$. Then, there exists a universal function K such that, for every $n \geqslant T$,*

$$\Big\| \sup_{t\in[0,T]} |X_t - \bar{X}_t^n| \Big\|_p \leq K(p,b,\sigma,T)\big(1+\|X_0\|_p\big)\left(\frac{T}{n}\right)^{\beta\wedge\frac{1}{2}}.$$

Proof I

- Still with $d = q = 1$, $b(t,x) = b(x)$, $\sigma(t,x) = \sigma(x)$ and $p = 2$.
- Combining the equations satisfied by X and its (continuous) Euler scheme yields

$$X_t - \bar{X}_t \;=\; \int_0^t \big(b(X_s) - b(\bar{X}_{\underline{s}})\big)\,ds + \int_0^t \big(\sigma(X_s) - \sigma(\bar{X}_{\underline{s}})\big)\,dW_s.$$

- Consequently, b and σ being Lipschitz, Schwartz's and Doob's Inequalities yield

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s - \bar{X}_s|^2\Big] &\leqslant 2\,\mathbb{E}\Big[\int_0^t [b]_{\rm Lip}|X_s - \bar{X}_{\underline{s}}|ds\Big]^2 + 2\,\mathbb{E}\Big[\sup_{s\in[0,t]}\Big(\int_0^s \big(\sigma(X_u) - \sigma(\bar{X}_{\underline{u}})\big)\,dW_u\Big)^2\Big]\\
&\leqslant 2\,[b]_{\rm Lip}^2\mathbb{E}\Big[\int_0^t |X_s - \bar{X}_{\underline{s}}|ds\Big]^2 + 8\,\mathbb{E}\int_0^t \big(\sigma(X_u) - \sigma(\bar{X}_{\underline{u}})\big)^2 du\\
&\leqslant 2t[b]_{\rm Lip}^2\int_0^t \mathbb{E}\,|X_s - \bar{X}_{\underline{s}}|^2 ds + 8\,[\sigma]_{\rm Lip}^2\int_0^t \mathbb{E}\,|X_u - \bar{X}_{\underline{u}}|^2 du.
\end{aligned}$$

Proof II

- Then

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s-\bar{X}_s|^2\Big]\leqslant C_{b,\sigma,T}\int_0^t \mathbb{E}\,|\,X_s-\bar{X}_{\underline{s}}|^2 ds$$
$$\leqslant C_{b,\sigma,T}\int_0^t \mathbb{E}\,|X_s-\bar{X}_s|^2 ds + C_{b,\sigma,T}\int_0^t \mathbb{E}\,|\bar{X}_s-\bar{X}_{\underline{s}}|^2 ds$$
$$\leqslant C_{b,\sigma,T}\int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]}|X_u-\bar{X}_u|^2\Big] ds + C_{b,\sigma,T}\int_0^t \mathbb{E}\,|\bar{X}_s-\bar{X}_{\underline{s}}|^2 ds.$$

- Let $f(t) := \mathbb{E}\left[\sup_{s\in[0,t]}|X_s-\bar{X}_s|^2\right]$, locally bounded by moment control Theorem.
- Gronwall's Lemma 8 at $t=T$ implies

$$\mathbb{E}\Big[\sup_{s\in[0,T]}|X_s-\bar{X}_s|^2\Big]\leqslant e^{C_{b,\sigma,T}T}C_{b,\sigma,T}\int_0^T \underbrace{\mathbb{E}\,|\bar{X}_s-\bar{X}_{\underline{s}}|^2}_{L^2(\mathbb{P})\text{-pathwise regularity of } s\mapsto \bar{X}_s} ds.$$

Proof II

- Now

$$\bar{X}_s - \bar{X}_{\underline{s}} = b(\bar{X}_{\underline{s}})(s-\underline{s}) + \sigma(\bar{X}_{\underline{s}})(W_s - W_{\underline{s}})$$

so that, using that $W_s - W_{\underline{s}}$ and $\bar{X}_{\underline{s}}$ are independent,

$$\mathbb{E}\,|\bar{X}_s - \bar{X}_{\underline{s}}|^2 \leqslant C_{b,\sigma}\left(\left(\frac{T}{n}\right)^2 \mathbb{E}\, b^2(\bar{X}_{\underline{s}}) + \mathbb{E}\,\sigma^2(\bar{X}_{\underline{s}})\,\mathbb{E}\,(W_s - W_{\underline{s}})^2\right)$$

by moment control Theorem (for the Euler scheme).

- Finally

$$\mathbb{E}\,|\bar{X}_s - \bar{X}_{\underline{s}}|^2 \leqslant C_{b,\sigma}\Big(1 + \mathbb{E}\sup_{t\in[0,T]}|\bar{X}_t|^2\Big)\left(\left(\frac{T}{n}\right)^2 + \frac{T}{n}\right)$$
$$\leqslant C_{b,\sigma}\big(1 + \mathbb{E}\,X_0^2\big)\frac{T}{n}.$$

L^2-Convergence rate: stepwise constant Euler scheme

Theorem (Stepwise versus genuine Euler scheme)

(a) *If b and σ satisfy the linear growth assumption with a real constant $L_{b,\sigma,T} > 0$, then, for every $p \in (0,+\infty)$ and every $n \geqslant T$,*

$$\left\| \sup_{t\in[0,T]} \Big| \bar{X}_t^n - \underbrace{\bar{X}_{\underline{t}}^n}_{=\widetilde{X}_t^n} \Big| \right\|_p \leq \tilde{\kappa}_p e^{\tilde{\kappa}_p L_{b,\sigma,T} T} \big(1 + \|X_0\|_p\big) \sqrt{\frac{T}{n}} \sqrt{1 + \log n}.$$

Proof

- Assume (for pure convenience) that $X_0 \in L^4$. One derives the linear growth assumption satisfied by b and σ (as Lipschitz functions) that

$$\sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}| \leqslant C_{b,\sigma}\big(1 + \sup_{t\in[0,T]} |\bar{X}_t|\big)\Big(\frac{T}{n} + \sup_{t\in[0,T]} |W_t - W_{\underline{t}}|\Big)$$

- so that

$$\Big\| \sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}|\Big\|_2 \leqslant C_{b,\sigma}\left\|\Big(1 + \sup_{t\in[0,T]} |\bar{X}_t|\Big)\left(\frac{T}{n} + \sup_{t\in[0,T]} |W_t - W_{\underline{t}}|\right)\right\|_2.$$

- Note that if U and V are two r.v., Schwarz's Inequality implies

$$\|U\,V\|_2 = \sqrt{\|U^2V^2\|_1} \leqslant \sqrt{\|U^2\|_2\|V^2\|_2} = \|U\|_4\|V\|_4.$$

- Combined with L^4-Minkowski inequality in both terms yields

$$\Big\| \sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}|\Big\|_2 \leqslant C_{b,\sigma}\Big(1 + \|\sup_{t\in[0,T]} |\bar{X}_t|\|_4\Big)\Big(\frac{T}{n} + \|\sup_{t\in[0,T]} |W_t - W_{\underline{t}}|\|_4\Big).$$

- Now, as already mentioned in the above "universal" bound,

$$\Big\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \Big\|_4 \leqslant C_W \sqrt{\frac{T}{n}} \sqrt{1+\log n}$$

- which completes the proof... if we admit that the L^4 moment control

$$\Big\| \sup_{t\in[0,T]} |\bar{X}_t| \Big\|_4 \leqslant C_{b,\sigma,T} \big(1 + \|X_0\|_4\big).$$

- (Almost) Finally

$$\Big\| \sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}| \Big\|_2 \leqslant C_{b,\sigma} \Big(1 + \|X_0\|\|_4\Big) \Big(\frac{T}{n} + \sqrt{\frac{T}{n}(1+\log n)}\Big).$$

- To switch from $\|X_0\|_4$ to $\|X_0\|_2$, one may first assume that X_0 is deterministic and then take the expectation of the square of this inequality with respect to the distribution $\mathbb{P}_{X_0}$ of X_0 (Exercise!). $\square$

- Combining the L^2-convergence rates of the genuine Euler scheme toward the diffusion in sup-norm and of $\sup_{t\in[0,T]}|\bar{X}^n_t - \widetilde{X}^n_t| = \sup_{t\in[0,T]}|\bar{X}^n_t - X^n_{\underline{t}}|$ yields the announce result by the triangle inequality.

Theorem (Stepwise constant Euler scheme *versus* diffusion ($p=2$))

(b) *In particular if b and σ satisfy Assumptions (H^β_T), for every $n \geqslant T$,*

$$\Big\|\sup_{t\in[0,T]}|X_t - \widetilde{X}^n_t|\Big\|_p \leqslant \widetilde{K}(p,b,\sigma,T)\big(1+\|X_0\|_p\big)\left[\sqrt{\frac{T(1+\log n)}{n}} + \left(\frac{T}{n}\right)^{\beta\wedge\frac{1}{2}}\right]$$

$$= O\left(\left(\frac{1}{n}\right)^{\beta} + \sqrt{\frac{1+\log n}{n}}\right)$$

where $\widetilde{K}(p,b,\sigma,T)$ is a universal function that can be determined.

- For the general case L^p-case, See the *Numerical Probability* book and/or a late edition of the note book .

A.s. convergence of the Euler scheme

Theorem

(a) If b and σ satisfy (H_T^β) for a $\beta \in (0,1]$ and if X_0 is a.s. finite, the continuous Euler scheme $\bar{X}^n = (\bar{X}_t^n)_{t\in[0,T]}$ a.s. converges toward the diffusion X for the sup*-norm over* $[0, T]$*.*
Furthermore,

$$\forall\, \alpha \in \left[0, \beta \wedge \tfrac{1}{2}\right), \qquad n^\alpha \sup_{t\in[0,T]} |X_t - \bar{X}_t^n| \xrightarrow{a.s.} 0.$$

(b) The same holds true for the stepwise constant Euler scheme.

Proof (Genuine Euler scheme)

- No *a priori* integrability assumption on X_0. But for simplicity we assume $X_0 \in \cap_{p>0} L^p(\mathbb{P})$.
- We know from L^p-convergence Theorem that, for every $p \geqslant 1$,

$$\forall\, n \geqslant 1,\ \mathbb{E}\Big(\sup_{t\in[0,T]} \big|\bar{X}^n_t - X_t\big|^p\Big) \leq C_{p,b,\sigma,\beta,T}\left(\frac{T}{n}\right)^{p(\beta\wedge\frac{1}{2})}\big(1+\|X_0\|_p\big)^p.$$

- Let $\alpha \in (0, \beta\wedge\frac{1}{2})$ and let $p > \frac{1}{\beta\wedge\frac{1}{2}-\alpha}$ so that $\displaystyle\sum_{n\geq 1}\frac{1}{n^{p(\beta\wedge\frac{1}{2}-\alpha)}} < +\infty$.
- Hence by Beppo Levi's monotone convergence Theorem for series

$$\mathbb{E}\Big(\sum_{n\geq 1} n^{p\alpha}\sup_{t\in[0,T]}\big|\bar{X}^n_t - X_t\big|^p\Big) \leqslant C_{p,b,\sigma,\beta,T}\sum_{n\geqslant 1} n^{-p(\beta\wedge\frac{1}{2}-\alpha)} < +\infty.$$

so that

$$\sum_{n\geq 1} n^{p\alpha}\sup_{t\in[0,T]}\big|\bar{X}^n_t - X_t\big|^p < +\infty \quad a.s.$$

Proof (end)

- Which in turn implies (by taking $\sqrt[p]{\cdot}$)

$$\mathbb{P}\text{-}a.s. \qquad \sup_{t\in[0,T]} |\bar{X}_t^n - X_t| = o\left(\frac{1}{n^\alpha}\right) \quad \text{as} \quad n \to +\infty. \qquad \square$$

- The proof for the stepwise constant Euler scheme follows exactly the same lines since an additional $\sqrt{1+\log n}$ term plays no role in the convergence of the above series.

- **Exercise**. Prove the result for a finite starting value X_0.

Question

- Is it possible to improve such strong L^p-rates achieved with the Euler-Maruyama schemes ?

One dimensional Milstein Scheme

- Motivation: higher order scheme to improve the strong rate of convergence.
- We consider the $SDE \equiv X_t = x + \int_0^t \sigma(X_s)dW_s$ for simplicity.
- Idea: expansion in small time $t \to 0$ taking advantage of σ differentiable. Itô's Lemma implies

$$\sigma(X_s) = \sigma(x) + \int_0^s \sigma'(X_u)dX_u + \frac{1}{2}\int_0^s \sigma''(X_u)d\langle X\rangle_u$$
$$= \sigma(x) + \int_0^s \sigma'\sigma(X_u)dW_u + \underbrace{\frac{1}{2}\int_0^s \sigma''(X_u)\sigma^2(X_u)du}_{=O_{L^2(\mathbb{P})}(s)}$$

$$\sigma(X_s) = \sigma(x) + \sigma\sigma'(x)W_s + O_{L^2(\mathbb{P})}(s)$$

since (Exercise!) $\left\| \int_0^s (\sigma\sigma(X_u) - \sigma\sigma'(x))dW_u \right\|_2 = O(s)$ if $\sigma\sigma'$ is Lipschitz.

so that

$$
\begin{aligned}
X_t &= x + \int_0^t \Big(\sigma(x) + \sigma\sigma'(x)W_s + O(s)\Big)\,dW_s \\
&= x + \underbrace{\sigma(x)W_t}_{\text{Euler term}} + \underbrace{\sigma\sigma'(x)\int_0^t W_s dW_s}_{\text{New term} \simeq O(t)} + \text{“}O(t^{\frac{3}{2}})\text{”}
\end{aligned}
$$

$$
X_t = x + \sigma(x)W_t + \frac{\sigma\sigma'(x)}{2}\Big(W_t^2 - t\Big) + \text{“}O(t^{\frac{3}{2}})\text{”}.
$$

- Combined with the propagation by the Markov property, this suggests to define the Milstein scheme as follows:

Definition (Discrete time Milstein scheme)

$$\bar{X}_0^{mil} = X_0,$$

$$\bar{X}_{t_{k+1}^n}^{mil} = \bar{X}_{t_k^n}^{mil} + \frac{T}{n} b(\bar{X}_{t_k^n}^{mil}) + \sigma(\bar{X}_{t_k^n}^{mil}) \sqrt{\frac{T}{n}}\, Z_{k+1}^n + \frac{1}{2}\sigma\sigma'(\bar{X}_{t_k^n}^{mil}) \frac{T}{n}\big((Z_{k+1}^n)^2 - 1\big)$$

where $Z_k^n = \sqrt{\frac{n}{T}}(W_{t_k^n} - W_{t_{k-1}^n}),\ k = 1, \ldots, n.$

or, equivalently, $\bar{X}_0^{mil} = X_0$

$$\bar{X}_{t_{k+1}^n}^{mil} = \bar{X}_{t_k^n}^{mil} + \frac{T}{n} b(\bar{X}_{t_k^n}^{mil}) + \sigma(\bar{X}_{t_k^n}^{mil}) \big(W_{t_{k+1}^n} - W_{t_k^n}\big) + \frac{1}{2}\sigma\sigma'(\bar{X}_{t_k^n}^{mil}) \Big((W_{t_{k+1}^n} - W_{t_k^n})^2 - \tfrac{T}{n}\Big),\ k = 0, \ldots, n-1.$$

- Or, by grouping the drift terms together,

Milstein scheme (bis)

$$\bar{X}^{mil}_{t^n_{k+1}} = \bar{X}^{mil}_{t^n_k} + \frac{T}{n}\left(b(\bar{X}^{mil}_{t^n_k}) - \frac{1}{2}\sigma\sigma'(\bar{X}^{mil}_{t^n_k})\right)$$
$$+\sigma(\bar{X}^{mil}_{t^n_k})\sqrt{\frac{T}{n}}\,Z^n_{k+1} + \frac{1}{2}\sigma\sigma'(\bar{X}^{mil}_{t^n_k})\frac{T}{n}(Z^n_{k+1})^2, \quad \bar{X}^{mil}_0 = X_0.$$

- **Remark.** If $\sigma = \sigma_0$ is constant then, $\sigma' \equiv 0$ so that

$$\bar{X}^{mil}_{t^n_k} = \bar{X}^{Euler}_{t^n_k}\ !$$

- **Example.** Black-Scholes model $dX_t = r\,X_t dt + \sigma X_t dW_t$, $X_0 = x_0 > 0$.

$$\begin{aligned}\bar{X}^{mil}_{t^n_{k+1}} &= \bar{X}^{mil}_{t^n_k} + r\bar{X}^{mil}_{t^n_k}\frac{T}{n} + \sigma\bar{X}^{mil}_{t^n_k}\Delta W_{t^n_{k+1}} + \frac{\sigma^2}{2}\bar{X}^{mil}_{t^n_k}\left((\Delta W_{t^n_{k+1}})^2 - \frac{T}{n}\right)\\ &= \bar{X}^{mil}_{t^n_k}\left(1 + (r - \frac{\sigma^2}{2})\frac{T}{n} + \sigma\Delta W_{t^n_{k+1}} + \frac{\sigma^2}{2}(\Delta W_{t^n_{k+1}})^2\right).\end{aligned}$$

Genuine Milstein scheme

- Like for Euler scheme the genuine Milstein scheme is obtained by letting the time and the Brownian motion evolve between two discretization instants while the coefficients b and σ remain frozen.
- The genuine Milstein scheme reads

$$\begin{aligned}
\bar{X}_t^{mil} &= \bar{X}_{\underline{t}}^{mil} + (t-\underline{t})\Big(b(\bar{X}_{\underline{t}}^{mil}) - \frac{1}{2}\sigma\sigma'(\bar{X}_{\underline{t}}^{mil})\Big) \\
&\quad +\sigma(\bar{X}_{\underline{t}}^{mil})(W_t - W_{\underline{t}}) + \frac{1}{2}\sigma\sigma'(\bar{X}_{\underline{t}}^{mil})(W_t - W_{\underline{t}})^2 \\
\bar{X}_0^{mil} &= X_0.
\end{aligned}$$

Theorem (Strong L^p-rate for the Milstein scheme)

Assume b and σ lie in $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ with bounded, $\alpha_{b'}$ and $\alpha_{\sigma'}$-Hölder continuous first derivatives respectively, $\alpha_{b'},\, \alpha_{\sigma'} \in (0,1]$.

(*a*) Discrete time and genuine Milstein scheme. *For every $p \in (0,+\infty)$, there exists a real constant $C_{b,\sigma,T,p} > 0$ such that, for every $X_0 \in L^p(\mathbb{P})$, independent of the Brownian motion W, one has*

$$\Big\|\max_{0\leqslant k\leqslant n}\big|X_{t_k^n}-\bar X^{mil,n}_{t_k^n}\big|\Big\|_p \leqslant \Big\|\sup_{t\in[0,T]}\big|X_t-\bar X^{mil,n}_t\big|\Big\|_p \leqslant C_{b,\sigma,T,p}\Big(\frac{T}{n}\Big)^{\frac{1+\alpha}{2}}\big(1+\|X_0\|_p\big)$$

where $\alpha = \min(\alpha_{b'},\alpha_{\sigma'})$.

(a') *In particular, if b' and σ' are (bounded) and Lipschitz continuous,*

$$\Big\|\max_{0\leqslant k\leqslant n}\big|X_{t_k^n}-\bar X^{mil,n}_{t_k^n}\big|\Big\|_p \leqslant \Big\|\sup_{t\in[0,T]}\big|X_t-\bar X^{mil,n}_t\big|\Big\|_p \leqslant C_{p,b,\sigma,T}\frac{T}{n}\big(1+\|X_0\|_p\big).$$

Good news for Euler !

Corollary

If $\sigma = \sigma_0 \neq 0$ and b is differentiable with a bounded Lipschitz derivative b', then (with obvious notations) $(\bar{X}^{eul}_t)_t = (\bar{X}^{mil}_t)_t$ so that, for every $p \in (0, +\infty)$,

$$\left\| X_T - \bar{X}^{eul,n}_T \right\|_p \leqslant \left\| \max_{0 \leqslant k \leqslant n} \left| X_{t^n_k} - \bar{X}^{eul,n}_{t^n_k} \right| \right\|_p$$
$$\leqslant \left\| \sup_{t \in [0,T]} \left| X_t - \bar{X}^{eul,n}_t \right| \right\|_p \leqslant C_{p,b,\sigma,T} \frac{T}{n} \big(1 + \|X_0\|_p\big).$$

...but not so good for the stepwise constant Misltein scheme!

Theorem (Strong L^p-rate for the stepwise constant Milstein scheme)

(b) Stepwise constant Milstein scheme. *As concerns the stepwise constant Milstein scheme*

$$(\widetilde{X}_t^{mil,n})_{t\in[0,T]} = (\bar{X}_{\underline{t}}^{mil,n})_{t\in[0,T]}$$

one has (like for, hence not better than, the Euler scheme!)

$$\Big\| \sup_{t\in[0,T]} \big|X_t - \widetilde{X}_t^{mil,n}\big| \Big\|_p \leqslant C_{b,\sigma,T}(1+\|X_0\|_p)\sqrt{\frac{T}{n}(1+\log n)}.$$

- Think to $X = W$ so that $\bar{X}_t^{mil} = W_t$ and $\bar{X}_{t_k^n}^{mil} = W_{t_k^n}$.