Yet other sequences: Kakutani's adding machine

$$\xi = \overline{0.a_1 a_2 \dots a_r \dots}^{\,p} \quad \text{regular } p\text{-adic expansion of } \xi \in [0,1].$$

- A new and very convenient addition: if $p = 10$

$$\begin{array}{cl} & 0.12333\dots \\ \oplus_{10} & 0.412777\dots \\ & \overline{} \\ = & 0.535011\dots \end{array}$$

- Let $y \in [\frac{1}{p}, 1] \cap \mathbb{Q}_p$ a *p*-adic rational number $\geqslant \frac{1}{p}$. Define the *p*-adic rotation with angle y by:

$$\forall\, x \in [0,1], \quad T_{p,y}(x) := x \oplus_p y.$$

- Let $p_1, \ldots, p_d$ the first d prime numbers. Let $y_i \in [\frac{1}{p}, 1] \cap \mathbb{Q}_p$, let $x_1, \ldots, x_d \in (0, 1]$. Set

$$\xi_n := \left(T^n_{p_i, y_i}(x_i)\right)_{1 \leqslant i \leqslant d}, \quad n \geqslant 1.$$

Theorem (Lapeyre-P., 1991)

The discrepancy of these sequence satisfy the uniform bound:

$$\forall\, n \geqslant 1,\ D^*_n(\xi_1, \ldots, \xi_n) \leqslant \frac{d - 1 + \prod_{i=1}^d (p_i - 1)\lfloor \log_{p_i}(n) \rfloor}{n} = O\left(\frac{(\log n)^d}{n}\right).$$

- **Remark.** If $y_i = x_i = \frac{1}{p_i}$ then $(\xi_n)_{n \geqslant 1} \equiv$ Halton.

- **Practitioner's corner**: Heuristic best choice $y_i = \frac{1}{p_i}$, $i = 1 : d$ and

$$x_i = \frac{1}{5} \text{ if } p_i \neq 5, 7,\ x_i = \frac{2p_i - 1 - \sqrt{(p_i + 1)^2 + 4p_i}}{3} \text{ if } p_i = 5 \text{ or } 7.$$

Faure sequence (1982)

- Let p be the smallest *prime* integer s.t. $p \geqslant d$. The d-dimensional Faure sequence is defined for every $n \geqslant 1$, by

$$\xi_n = \Big(\Phi_p(n-1), C_p\big(\Phi_p(n-1)\big), \cdots, C_p^{d-1}\big(\Phi_p(n-1)\big)\Big)$$

where for every $u \in \mathbb{Q}_p$ (p-adic rational) $u = \sum_{k\geqslant 0} u_k p^{-(k+1)} \in [0,1]$

$$C_p(u) = \sum_{k\geqslant 0} \underbrace{\left(\sum_{j\geqslant k} \binom{j}{k} u_j \mod. p\right)}_{\in\{0,\ldots,p-1\}} p^{-(k+1)}.$$

- Its discrepancy at the origin (Faure, 1982)

$$D_n^*(\xi) \leqslant \frac{1}{n}\left(\frac{1}{d!}\left(\frac{p-1}{2\log p}\right)^d (\log n)^d + O((\log n)^{d-1})\right).$$

A new hope?

- The prominent feature of Faure's estimate is that the the leading error coefficient satisfies
$$\lim_{d} \frac{1}{d!}\left(\frac{p-1}{2\log p}\right)^d = 0$$
which suggests an (almost) better rate than $O\left(\frac{(\log n)^d}{n}\right)$ as $d \to +\infty$.
[The above convergence result is an easy consequence of Stirling's formula and Bertrand's conjecture]

- To be more precise, the function $x \mapsto \frac{x-1}{\log x}$ being increasing on $(0, +\infty)$,
$$\frac{1}{d!}\left(\frac{p-1}{2\log p}\right)^d \leqslant \frac{1}{d!}\left(\frac{2d-1}{2\log 2d}\right)^d \precsim \left(\frac{e}{\log 2d}\right)^d \frac{1}{\sqrt{2\pi d}} \to 0 \quad \text{as} \quad d \to +\infty.$$

The "$P_{p,d}$" property I

- An unpublished non-asymptotic upper-bound due to Y.J. Xiao in his PhD thesis is

$$\forall\, n \geqslant 1, \quad D_n^*(\xi) \leqslant \frac{1}{n}\left(\frac{1}{d!}\left(\frac{p-1}{2}\right)^d\left(\left\lfloor\frac{\log(2n)}{\log p}\right\rfloor + d + 1\right)^d\right).$$

t

- Same leading coefficient but... if $d = p = 5$ and $n = 1\,000$,

$$D_n^*(\xi) \leqslant \frac{1}{n}\left(\frac{1}{d!}\left(\frac{p-1}{2}\right)^d\left(\left\lfloor\frac{\log(2n)}{\log p}\right\rfloor + d + 1\right)^d\right) \approx 1.18$$

which is of little interest since $D_n^*(\xi) \in [0,1]$ by construction.

- Look at the "constant" term (in the $(\log n)^k$-scale):

$$\lim_{d\to+\infty} \frac{(d+1)^d}{d!}\left(\frac{p-1}{2}\right)^d = +\infty$$

The "$P_{p,d}$" property II

- It has been shown later on by Niederreiter that Faure's sequences satisfy the so-called "$P_{p,d}$" property

Proposition (Niederreiter)

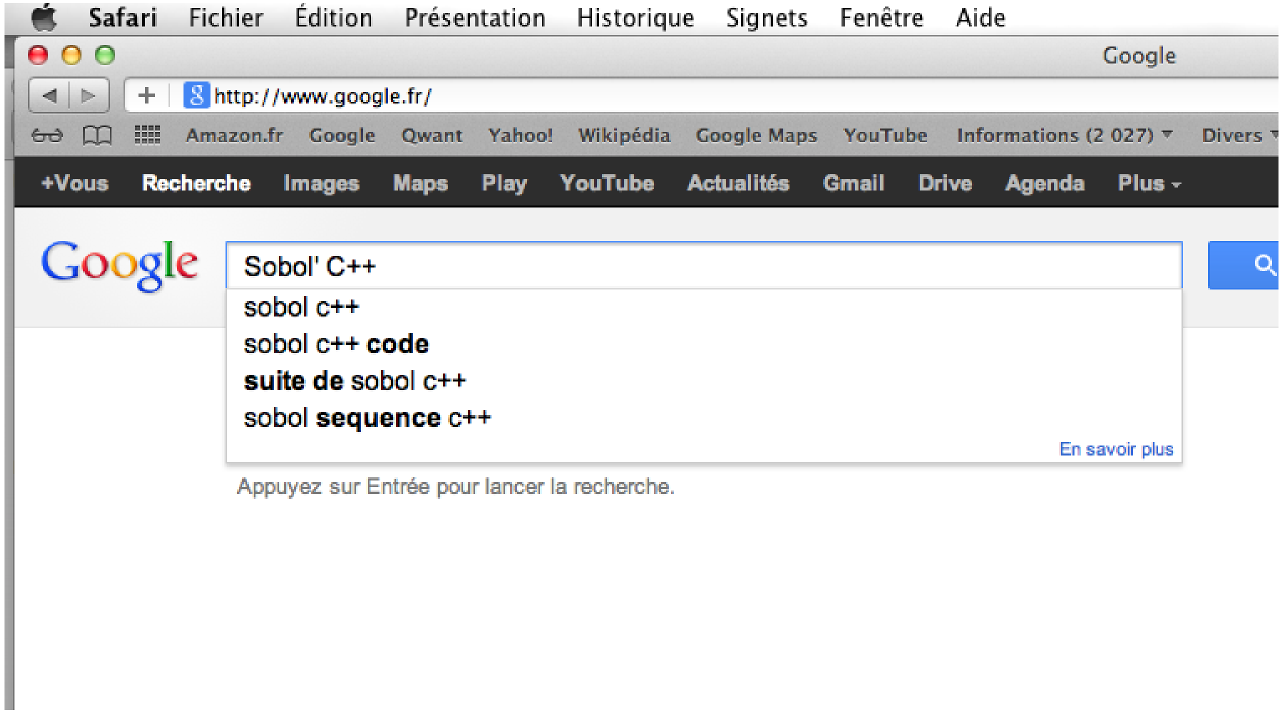
Let $m \in \mathbb{N}$ and let $\ell \in \mathbb{N}^*$. For any $r_1, \ldots, r_d \in \mathbb{N}$ such that $r_1 + \cdots + r_d = m$ and every $x_1, \ldots, x_d \in \mathbb{N}$, there is exactly one term in the sequence $\xi_{\ell p^m + i}$, $i = 0, \ldots, p^m - 1$ lying in the hyper-cube

$$\prod_{k=1}^{d} \left[x_k p^{-r_k}, (x_k + 1) p^{-r_k}\right).$$

- After Halton's sequences but before Faure's and Niederreiter's sequences ...

Sobol' sequences

- Sobol' sequence (in the modified version by Antonov and Saleev)



- Optimized code using Gray code (almost unrivaled in practice).
- Initializations of the "directions" remains a difficult question
- Generalization/systemization of both Faure and Sobol' sequence by Niederreiter in connection with finite fields.

MC *vs* QMC II

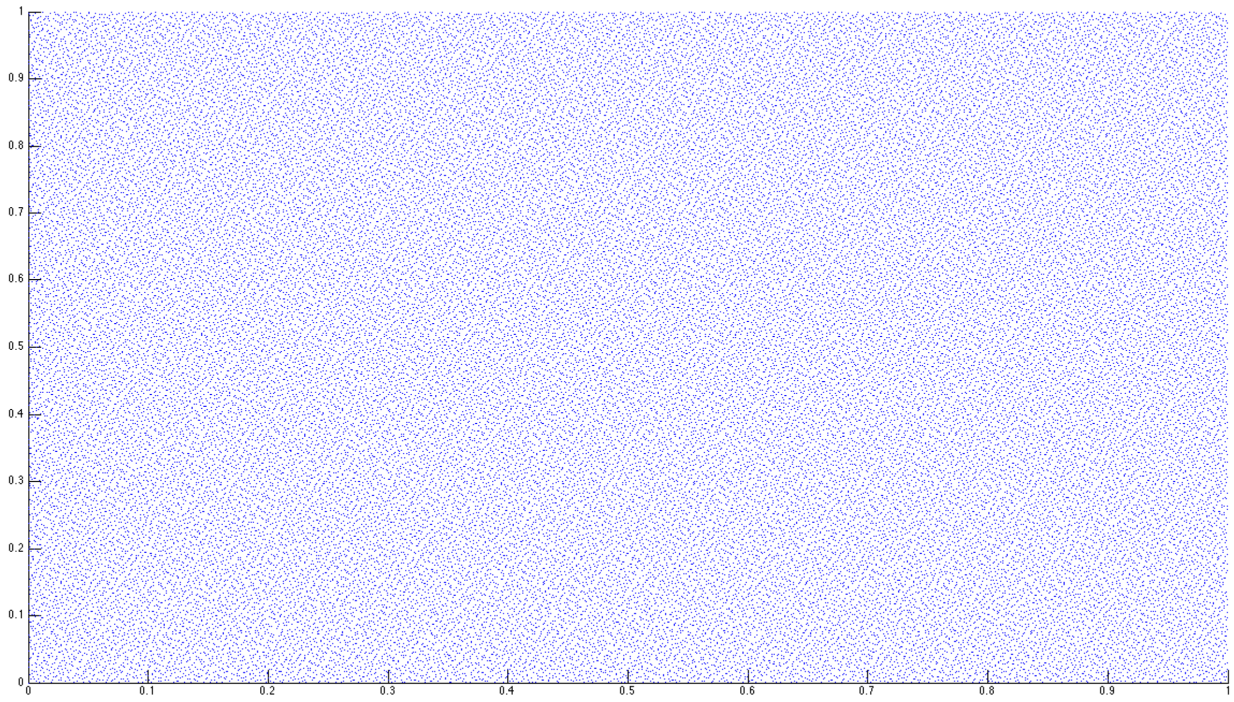


Figure: $n = 60\,000$ but $\ldots$ MC or QMC?

MC *vs* QMC II

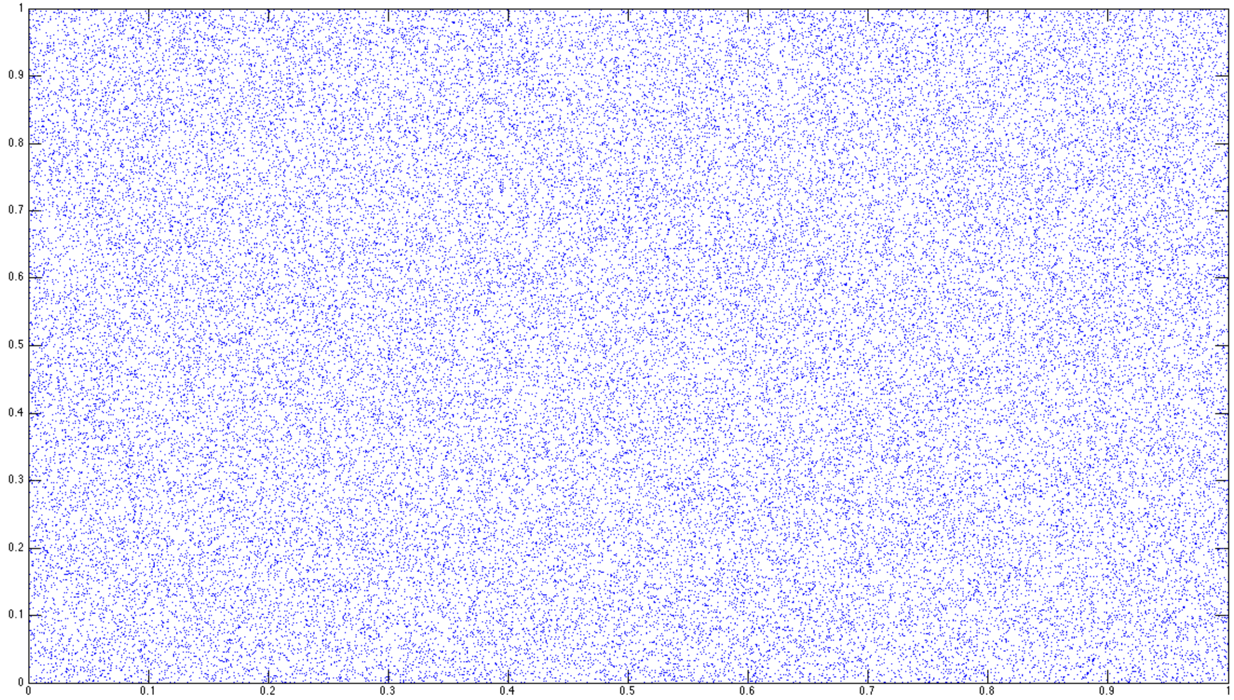


Figure: $n = 60\,000$ but... MC or QMC?

MC *vs* QMC II (bis)

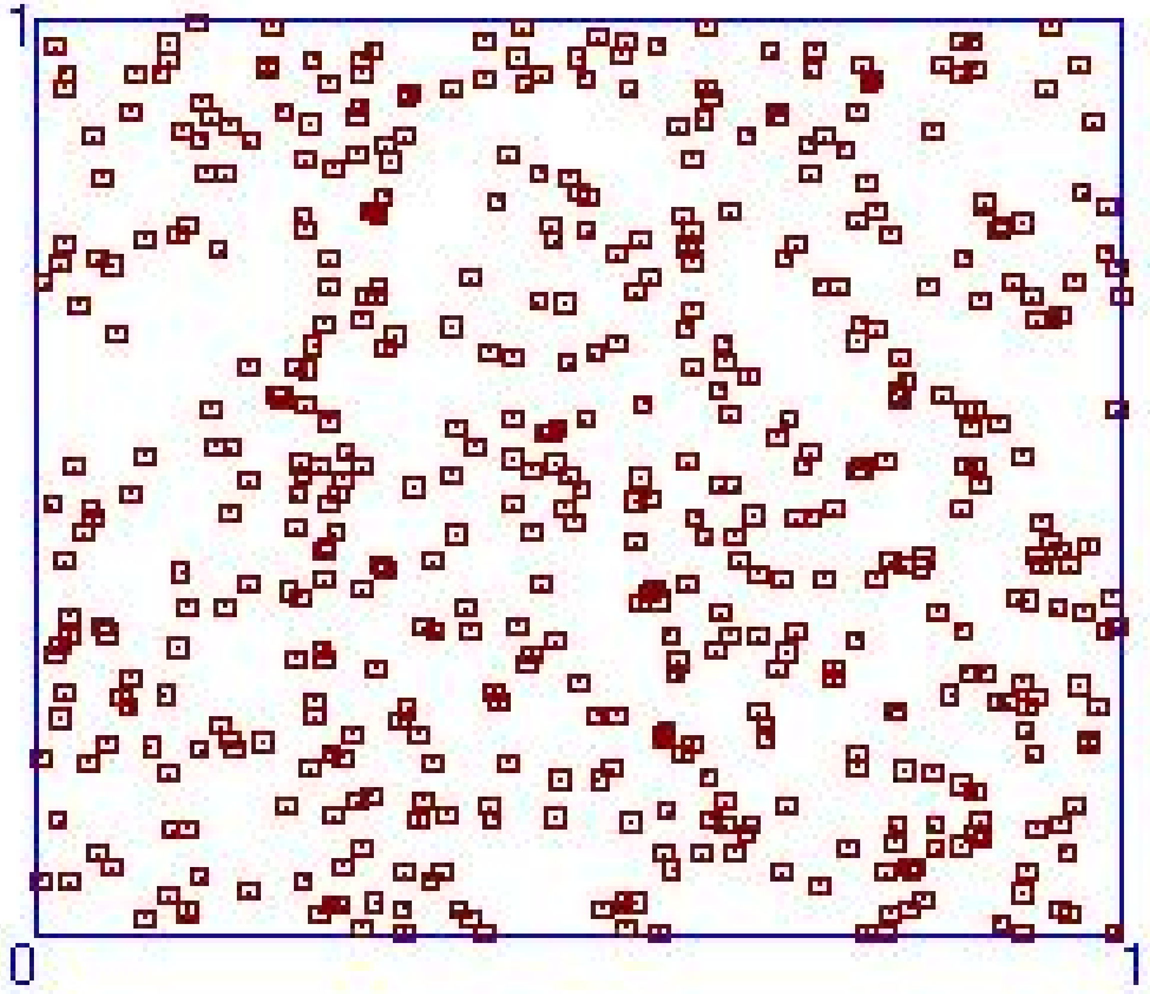
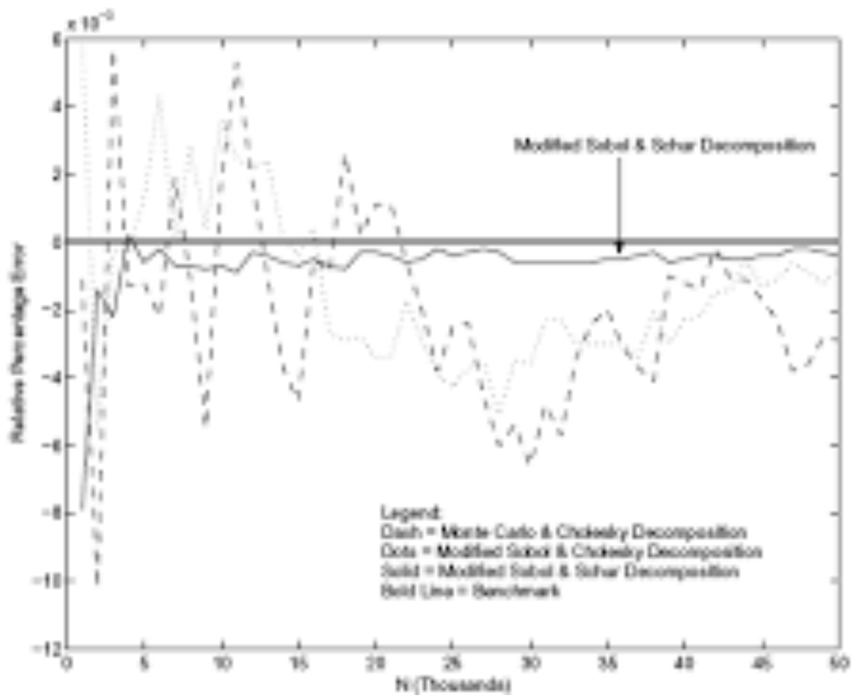


Figure: Pseudo-random numbers $n = 500$: MC.

Application: Geometric mean option

- A 2D-Black-Scholes model: $dX_t^i = X_t^i(rdt + \sigma_i dW_t^i)$, $i = 1, 2$, $\langle W^1, W^2\rangle_t = \rho t$, ρß$n[_1, 1]$.
- Geometric mean payoff $h_T = \left(\sqrt{X_T^1 X_T^2} - K\right)^+$ (closed form).

Figure 2 – Error of the geometric average option



The curse of dimensionality strikes back

Theorem (Proinov (1991))

Let $\mathbb{R}^d$ be equipped with the ℓ^∞-norm ($|(x^1,\ldots,x^d)|_\infty := \max_{1\leqslant i\leqslant d}|x^i|$).

(a) Let $(\xi_1,\ldots,\xi_n)\in([0,1]^d)^n$. For every continuous function $f:[0,1]^d\to\mathbb{R}$,

$$\left|\int_{[0,1]^d} f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(\xi_k)\right| \leqslant C_d\, w\Big(f, D_n^*(\xi_1,\ldots,\xi_n)^{\frac{1}{d}}\Big)$$

where $w(f,\delta) := \sup_{x,y\in[0,1]^d,\,|x-y|_\infty\leqslant\delta}|f(x)-f(y)|$, $\delta\in(0,1)$ is the uniform continuity modulus of f (for the ℓ^∞-norm).

(b) If $d=1$, $C_d=1$ and if $d\geqslant 2$, $C_d\in[1,4]$.

Theorem (Proinov (1991))

(c) *In particular, if f is Lipschitz continuous with ℓ^∞-Lipschitz coefficient* $[f]_{\mathrm{Lip}}$,

$$\left|\int_{[0,1]^d} f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(\xi_k)\right| \leqslant C_d\,[f]_{\mathrm{Lip}} D_n^*(\xi_1,\dots,\xi_n)^{\frac{1}{d}}.$$

- Hence, the best (known) possible universal rate for a *sequence with low discrepancy* along Lipschitz functions is then

$$\sup_{f:[f]_{\mathrm{Lip}}\leqslant 1}\left|\int_{[0,1]^d} f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(\xi_k)\right| = O\left(\frac{\log n}{n^{\frac{1}{d}}}\right)$$

- Equivalently, using the L^1-Wasserstein distance:

$$\mathcal{W}_1\Big(\frac{1}{n}\sum_{1\leqslant k\leqslant n}\delta_{\xi_k},\lambda_{d|[0,1]^d}\Big) = O\left(\frac{\log n}{n^{\frac{1}{d}}}\right).$$

Lipschitz *versus* finite variation on $[0,1]^d$

- If $d = 1$ Lipschitz continuous $\Longrightarrow$ finite variation and $V(f) \leqslant [f]_{\rm Lip}$.
- If $d \geqslant 2$, it is no longer true.
- The functions $f_d : [0,1]^d \to \mathbb{R}$ defined by

$$f_d(\xi^1, \ldots, \xi^d) = (\xi^1 + \cdots + \xi^d) \wedge 1$$

  have finite variation iff $d = 1, 2$.
- The Box-Muller transform(s) $(\xi^1, \xi^2) \mapsto \left(\sqrt{-2\log(\xi^1)}\sin(2\pi\xi^2)\right)$ and/or $(\xi^1, \xi^2) \mapsto \left(\sqrt{-2\log(\xi^1)}\cos(2\pi\xi^2)\right)$ have infinite variation on $[0,1]^2$...
- Space of functions with finite variation become sparse in $\mathcal{B}or([0,1]^d, \mathbb{R})$.
- Numerical experiments show *a posteriori* better performance than predicted by Proinov's Theorem on functions with infinite variation.
- But no confidence interval ... In fact no error control!

WARNING ! Prohibited use cases. . .

- Is there a price to pay for filling up the hypercube faster than random numbers ?
- YES! A single component DOES NOT EMULATE INDEPENDANCE
- Practionner's viewpoint: Van der Corput sequence Vdc(2): satisfies

$$\xi_{2n} = \frac{\xi_n}{2}, \quad \xi_{2n+1} = \xi_{2n} + \frac{1}{2} = \frac{\xi_n + 1}{2}, \qquad n \geqslant 1.$$

Then

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \xi_{2k}\xi_{2k+1} &= \frac{1}{4n}\sum_{k=1}^{n} \xi_k(\xi_k + 1) \\ &\to \frac{1}{4}\int_0^1 x(x+1)dx \\ &= \frac{5}{24} \neq \left(\frac{1}{2}\right)^2. \end{aligned}$$

New horizons?

- After a long deny, the *QMC* community admitted the curse of dimensionality.
- This led to devise Randomized QMC where *QMC* appears as a control variate.