δ-hedge

- Black-Scholes dynamics

$$X_t^x = x\, e^{(r-\frac{\sigma^2}{2})t+\sigma W_t},\ t \in [0, T].$$

- $\Phi(x) = \mathbb{E}\,\varphi(X_T^x)$ (forget about the discount factor).

Proposition (Sensitivity δ-hedge)

(a) If $\varphi : (0, +\infty) \to \mathbb{R}$ is locally Lipschitz, differentiable outside a countable set such that φ' has polynomial growth, then Φ is differentiable and

$$\forall\, x > 0, \quad \Phi'(x) = \mathbb{E}\left(\varphi'(X_T^x)\frac{X_T^x}{x}\right).$$

(b) If $\varphi : (0, +\infty) \to \mathbb{R}$ is simply a Borel function with polynomial growth, then Φ is still differentiable and

$$\forall\, x > 0, \quad \Phi'(x) = \mathbb{E}\left(\varphi(X_T^x)\frac{W_T}{x\,\sigma\,T}\right).$$

Proof I: Stein's identity

- Let $Z \sim \mathcal{N}(0,1)$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable g' has polynomial growth (hence g has too). Then

$$\mathbb{E}\, g'(Z) = \mathbb{E}\, g(Z)Z$$

In fact, using an integration by parts

$$\begin{aligned}
\mathbb{E}\, g'(Z) &= \int_{\mathbb{R}} g'(z)\, e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{2\pi}} \big[g(z)\, e^{-z^2/2}\big]_{z=-\infty}^{z=+\infty} - \int_{-\infty}^{+\infty} g(z)\big(-z\, e^{-z^2/2}\big) \frac{dz}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{+\infty} g(z) z\, e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \\
&= \mathbb{E}\, g(Z)Z.
\end{aligned}$$

Proof II (Identity for a smooth function)

- One has, if φ is smooth

$$\Phi'(x) = \int_{\mathbb{R}} \varphi'\big(x\exp\big(\mu T + \sigma\sqrt{T}z\big)\big)\exp\big(\mu T + \sigma\sqrt{T}z\big)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}$$
$$= \frac{1}{x\,\sigma\sqrt{T}}\mathbb{E}g'(Z)$$

with

$$g(z) = \varphi\big(x\exp\big(\mu T + \sigma\sqrt{T}z\big)\big).$$

- Hence, by Stein's identity

$$\Phi'(x) = \frac{1}{x\,\sigma\sqrt{T}}\mathbb{E}\,g(Z)Z$$
$$= \frac{1}{x\,\sigma\sqrt{T}}\mathbb{E}\left[\varphi\big(x\exp\big(\mu T + \sigma\sqrt{T}Z\big)\big)Z\right]$$
$$= \frac{1}{x\,\sigma\,T}\mathbb{E}\left[\varphi(X_T^x)W_T\right].$$

where we came back on the scenario space Ω in the last line.

Proof III (Regularization)

- Note that the above identity does not involve the derivative φ',
- By a regularization/approximation process of φ by differentiable functions, one shows that the equality still holds for any Borel function φ with polynomial growth.

δ-hedge & variance reduction (the Call case)

- These estimators can be improved (at least for short maturities) by introducing control variates as follows:

$$\Phi'(x) = \varphi'(x) + \mathbb{E}\left[\left(\varphi'(X_T^x) - \varphi'(x)\right)\frac{X_T^x}{x}\right] = \mathbb{E}\left[\left(\varphi(X_T^x) - \varphi(x)\right)\frac{W_T}{x\sigma T}\right].$$

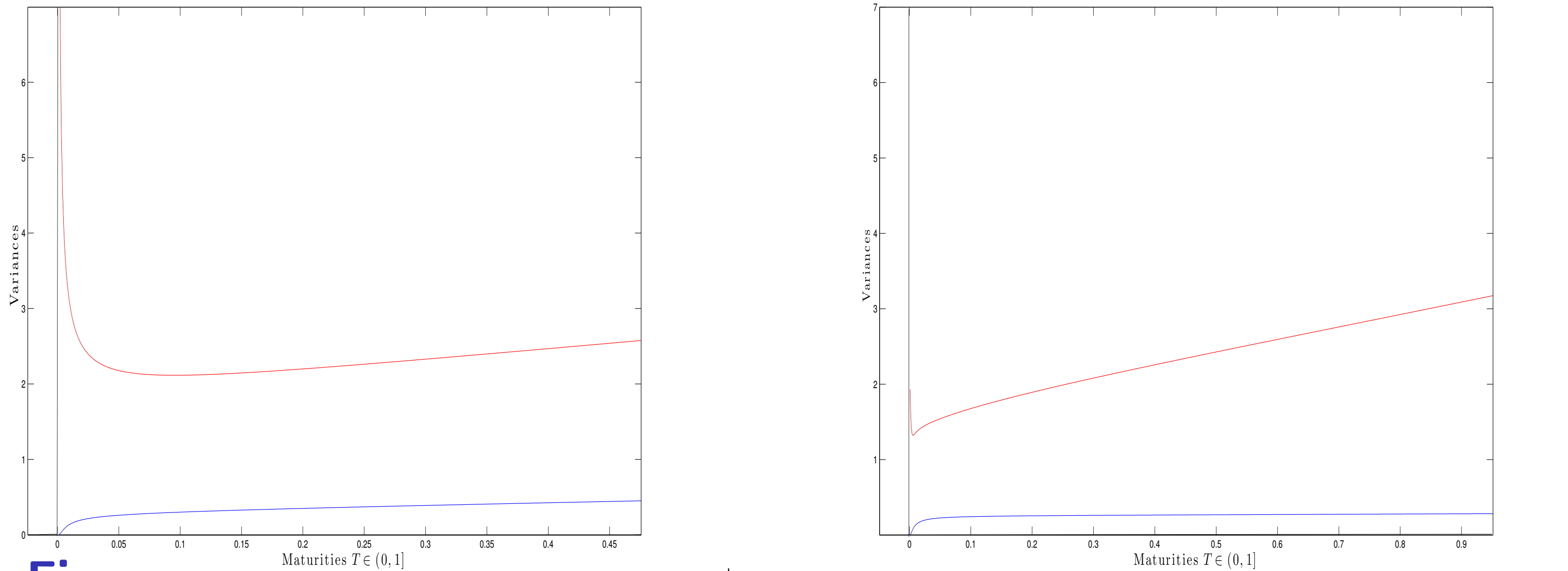
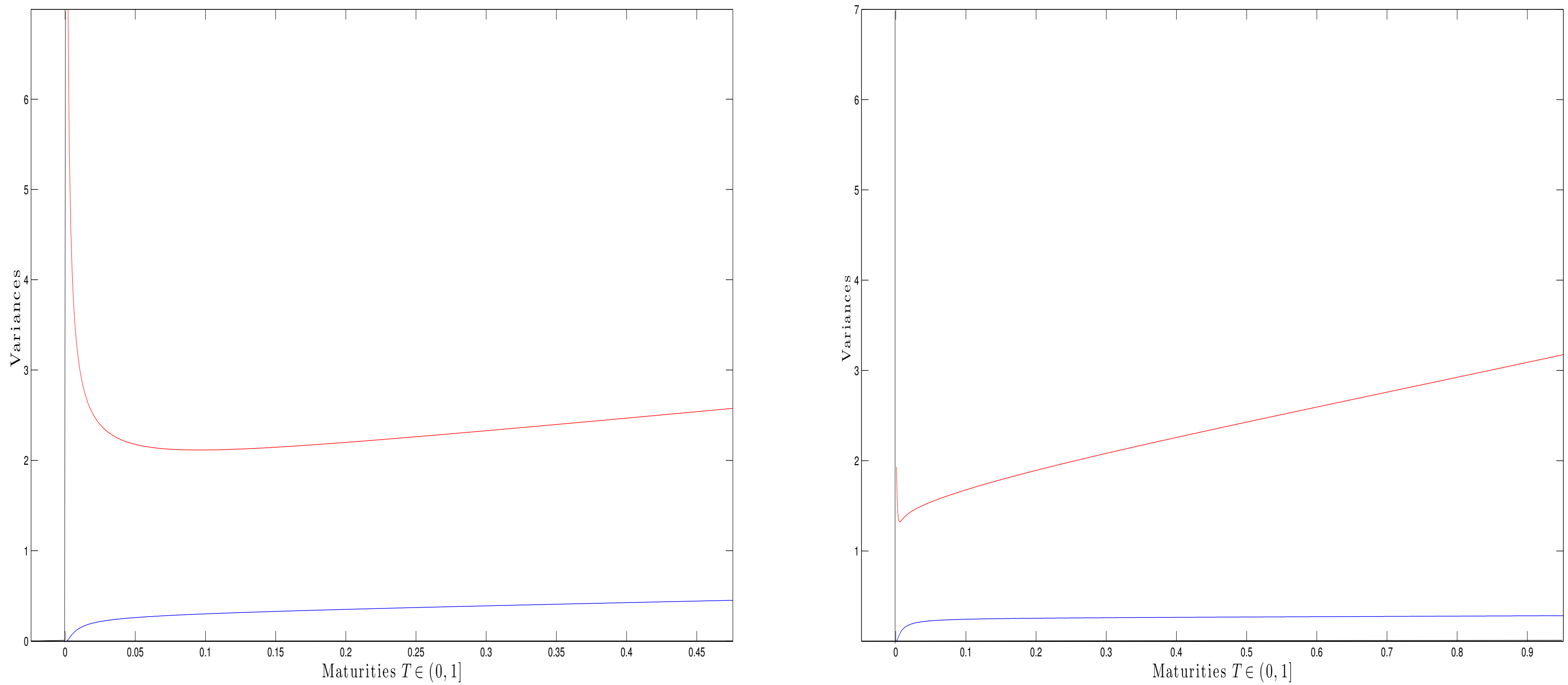


Figure: Black-Scholes Call $\varphi(x) = (x-K)^+$: $x = 100$, $K = 95$, $r = 0$, $\sigma = 0.5$, $T \in (0,1]$. *Left:* $T \mapsto \mathrm{Var}\left(\varphi'(X_T^x)\frac{X_T^x}{x}\right)$ *(blue line)* and $T \mapsto \mathrm{Var}\left(\varphi(X_T^x)\frac{W_T}{x\sigma T}\right)$ *(red line)*, *Right:* $T \mapsto \mathrm{Var}\left(\left(\varphi'(X_T^x) - \varphi'(x)\right)\frac{X_T^x}{x}\right)$ *(blue line)* and $T \mapsto \mathrm{Var}\left(\left(\varphi(X_T^x) - \varphi(x)\right)\frac{W_T}{x\sigma T}\right)$ *(red line)*.



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From Monte Carlo to . . .

Quasi-Monte Carlo

Basic of Monte Carlo simulation (background!)

- **Simulation**:

$$X = \varphi(U), \quad U = (U^1, \ldots, U^d) \stackrel{d}{=} U\big([0,1]^d\big),\ d \in \mathbb{N}^*.$$

[Rejection method $d = (\mathbb{N}^*)$ and $X = \varphi(\tau, U_1, \ldots, U_\tau)$, $\tau(\mathcal{F}_n^u)$-stopping time.]

- *SLLN*: $X = \varphi(U) \in L^1$,

$$X_k = \varphi(U_k),\ k \geqslant 1,\ i.i.d.$$

then (be careful: we switch from, M to n)

$$\bar{X}_n := \frac{1}{n}\sum_{k=1}^{n} \varphi(U_k) \stackrel{n\to+\infty}{\longrightarrow} \mathbb{E}X = \int_{[0,1]^d} \varphi(u)du^1 \cdots du^d \ (a.s. \,\&\, L^1).$$

- *CLT*: $X = \varphi(U) \in L^2$,

$$\sqrt{n}\Big(\bar{X}_n - \mathbb{E}X\Big) \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{N}(0; \sigma_X^2)$$

where $\sigma_X^2 = \mathbb{E}(X - \mathbb{E}\,X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$.

Sacrifice(s)?

- What are we ready to sacrifice to speed up the rate of convergence $\sqrt{n}$?
  - The absence of regularity of φ?
  - The flexibility induced by the independence of successive terms ?
  - The data-driven control of the error (confidence interval induced by the *CLT*?
  - ALL THAT ?
- Let us go deeper in the exploration of the rate of convergence of the *SLLN*.

Monte Carlo method through empirical measure

Theorem (Glivenko-Cantelli & Kolmogorov-Smirnov)

(*a*) *Let* $(U_k)_{k\geqslant 1}$ *be i.i.d. with* $\mathcal{U}([0,1]^d)$*-distribution.*

$$\mathbb{P}(d\omega)\text{-a.s.}\ \left(\forall\, f \in \mathcal{C}_{basc}([0,1]^d,\mathbb{R})\ \ \frac{1}{n}\sum_{k=1}^{n} f\big(U_k(\omega)\big) \overset{n\to\infty}{\longrightarrow} \int_{[0,1]^d} f(\xi)d\xi\right).$$

where $\mathcal{C}_{basc}([0,1]^d,\mathbb{R}) = \big\{f : [0,1]^d \to \mathbb{R} : \textit{bounded, } d\xi\textit{-a.s. continuous}\big\}$.

(*b*) Discrepancy at the origin*: let* $x := (x^1,\ldots,x^d)\in[0,1]^d$, $[\![0,x]\!] = \prod_{i=1}^{d}[0,x^i]$.

$$D_n^*(U_1,\ldots,U_n) := \sup_{x\in[0,1]^d}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{U_k\in[\![0,x]\!]\}} - \mathrm{Vol}([\![0,x]\!])\right| \overset{n\to+\infty}{\longrightarrow} 0\ a.s.$$

Remark. $D_n^*(U_1,\ldots,U_n)(\omega)\in[0,1]$ is called Kolmogorov-Smirnov distance between $\mu_n^U(\omega,d\xi)$ and $\lambda_{d|[0,1]^d}$.

Theorem (Chung)

(a) *Chung's CLT*:

$$\sqrt{n}\, D_n^*(U_1,\ldots,U_n) \xrightarrow{\mathcal{L}} \sup_{x\in[0,1]^d} |Z^x|$$

where $(Z^x)_{x\in[0,1]^d}$ *is a Gaussian process with covariance structure*

$$\mathrm{Cov}(Z^x, Z^y) = \prod_{i=1}^{d}(x^i \wedge y^i) - \prod_{i=1}^{d} x^i \prod_{i=1}^{d} y^i,$$

(bridge of the hyper Brownian sheet).

(b) *Chung's LIL:*

$$\overline{\lim_n} \sqrt{\frac{2n}{\log\log n}}\, D_n^*(U_1,\ldots,U_n) = 1.$$

Uniformly distributed sequences

Idea! Find deterministic sequences $(\xi_n)_{n\geqslant 1}$ whose star discrepancy

$$D_n^*(\xi_1,\ldots,\xi_n) = \sup_{x\in[0,1]^d}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{\xi_k\in[\![0,x]\!]\}} - \mathrm{Vol}([\![0,x]\!])\right| \longrightarrow 0 \text{ as } n\to+\infty$$

(note that $\forall\, n\geqslant 1,\ D_n^*(\xi_1,\ldots,\xi_n)\in[0,1]$ by construction)
Consequently

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\xi_k}\overset{weakly}{\longrightarrow}\lambda_{d|[0,1]^d}$$

(Lebesgue measure on $[0,1]^d=\mathcal{U}([0,1]^d)$) which implies

$$\boxed{(*)\quad \forall\, f\in\mathcal{C}([0,1]^d,\mathbb{R}),\quad \lim_n\frac{1}{n}\sum_{k=1}^{n}f(\xi_k)=\int_{[0,1]^d}f(u)du.}$$

$(*)$ Extends to bounded λ_d-*a.s.* continuous Borel functions f (like indicator of hyper-rectangles).

In order to carry on the comparison with confidence intervals of the Monte Carlo method, the question is

▷ What about error bounds to control the numerical integration error ?

Koksma-Hlawka Inequality...

Let $(\xi_n)_{n\geqslant 1}$ be any $[0,1]^d$-valued sequence.

Theorem (Koksma-Hlawka, 1943(!))

If $f : [0,1]^d \to \mathbb{R}$ *has finite variation in the measure sense i.e.*

$$f(x) = f(\mathbf{1}) + \nu([\![0, 1-x]\!]), \quad \nu \ \textit{signed measure}, \ \nu(0) = 0$$

then

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^d} f(u)du\right| \leqslant V(f) \times D_n^*(\xi_1, \ldots, \xi_n)$$

where

$$V(f) := |\nu|([0,1]^d) = |\nu|([0,1]^d)$$

denotes the variation of f.

Remark. There exists amore general notion of function with finite variation known as " Hardy & Krause sense".

A “child” version. . .

If $d = 1$ and f is differentiable with $\displaystyle\int_0^1 |f'| d\lambda_1 < +\infty$, then f has finite variation and $V(f) = \int_0^1 |f'| d\lambda_1$ so that

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]} f(u)du\right| \leqslant \int_0^1 |f'| d\lambda_1 \times D_n^*(\xi_1, \ldots, \xi_n).$$

Well, but what is it for?...

Theorem (Low discrepancy sequences and n-tuples)

(a) *There exists* $[0,1]^d$*-valued sequences* $(\xi_n)_{n\geqslant 1}$ *such that*

$$\forall\, n \geqslant 2, \qquad D_n^*(\xi_1,\dots,\xi_n) \leqslant C(\xi)\frac{(\log n)^d}{n}$$

(b) *Hammersley's principle: If we set for such a sequence*

$$\widetilde{\xi}_k^n = \Big(\xi_k, \frac{k}{n}\Big) \in [0,1]^{d+1},$$

then, there exists a real constant $\widetilde{C}(\xi)$ *only depending on* $C(\xi)$ *s.t.*

$$\forall\, n \geqslant 2, \qquad D_n^*(\widetilde{\xi}_1^n,\dots,\widetilde{\xi}_n^n) \leqslant \widetilde{C}(\xi)\frac{(\log n)^d}{n}.$$

- Recursive numerical integration is possible at rate $O\left(\frac{(\log n)^d}{n}\right)$.

- Static numerical integration is possible at rate $O\left(\frac{(\log n)^{d-1}}{n}\right)$.

- When $d = 1$ if $(\xi_1^{(n)}, \ldots, \xi_n^{(n)})$ denotes the reordering of the n-tuple $(\xi_1, \ldots, \xi_n)$

$$D_n^*(\xi_1, \ldots, \xi_n) = \frac{1}{2n} + \max_{1 \leqslant k \leqslant n} \left| \xi_k^{(n)} - \frac{2k-1}{2n} \right|$$

so that the minimal discrepancy is obtained at mid-points:

$$\xi_k = \frac{2k-1}{2n}, \; k = 1 : n.$$

and

$$D_n^*\left(\frac{2k-1}{2n}, \; k = 1 : n\right) = \frac{1}{2n}.$$

Halton and Van der Corput Sequences

- $d = 1$. Let $p \in \mathbb{N}$, $p \neq 1$. The Radical inverse p-transform Φ_p:

$$\begin{aligned} n &= a_r p^r + \cdots + a_0,\ a_i \in \{0, \ldots, p-1\} \\ &\downarrow \\ \Phi_p(n) &= \frac{a_r}{p^{r+1}} + \cdots + \frac{a_0}{p}. \end{aligned}$$

Theorem (Van der Corput)

For every $n \geqslant 1$, $\xi_n = \Phi_p(n)$. Then

$$\forall\, n \geqslant 1, \quad D_n^*(\xi_1, \ldots, \xi_n) \leqslant \frac{(p-1)\lfloor \log_p(n) \rfloor}{n} = O\left(\frac{\log n}{n}\right)$$

and $\underline{\lim}_n\, n D_n^*(\xi_1, \ldots, \xi_n) = 1$.

- *E.g.*, if $p = 2$, the first terms of the $VdC(2)$ sequence read

$$\xi_1 = \frac{1}{2},\ \xi_2 = \frac{1}{4},\ \xi_3 = \frac{3}{4},\ \xi_4 = \frac{1}{8},\ \xi_5 = \frac{5}{8},\ \xi_6 = \frac{3}{8},\ \xi_7 = \frac{7}{8}, \ldots$$

- $d \geqslant 2$. It is the d-dimensional extension. It relies on the Chinese remainder theorem.

Theorem (Halton)

Let $p_1, \ldots, p_d$ the first d prime numbers. Let

$$\xi_n = (\Phi_{p_1}(n), \ldots, \Phi_{p_d}(n)),\ n \geqslant 1.$$

$$D_n^*(\xi_1, \ldots, \xi_n) \leqslant \frac{1 + \prod_{i=1}^d (p_i - 1)\lfloor \log_{p_i}(n) \rfloor}{n} = O\left(\frac{(\log n)^d}{n}\right).$$