Cox-Ingersol-Ross process (CIR)

- The *CIR* process is solution to the *SDE*

$$dX_t = \kappa(a - X_t)dt + \vartheta\sqrt{X_t}dW_t, \quad X_0 > 0$$

where $\kappa, a, \vartheta > 0$.

- Though $\sqrt{x}$ is not Lipschitz (even defined) on $\mathbb{R}$...

Theorem (Strong solution for the *CIR*, Ikeda-Watanabe, 1981)

If $\frac{\vartheta^2}{2\kappa a} \leqslant 1$, for every $x > 0$, there exists a unique solution $(X_t^x)_{t\geqslant 0}$ and

$$\mathbb{P}\text{-a.s.} \quad X_0^x = x \quad \text{and} \quad \forall\, t \geqslant 0, \quad X_t^x > 0.$$

- Naive (formal) Euler discretization scheme:

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \kappa(a - \bar{X}_{t_k^n})\frac{T}{n} + \vartheta\sqrt{\bar{X}_{t_k^n}}\Delta W_{t_{k+1}^n},\ k = 0 : n-1,\ \bar{X}_0 = x$$

is not valid since $\mathbb{P}(\bar{X}_{t_1^n} < 0) = \mathbb{P}\big(W_{T/n} < z(x)\big) > 0$.

- Can it be fixed by replacing $\sqrt{\bar{X}_{t_k^n}}$ by $\sqrt{|\bar{X}_{t_k^n}|}$ *mutatis mutandis* ? Implementable, but still not positive.

Implicit scheme I

- Naive (semi-) implicit scheme (to preserve martingality):

$$\bar{X}_{t^n_{k+1}} = \bar{X}_{t^n_k} + \kappa(a - \bar{X}_{t^n_{k+1}})\frac{T}{n} + \vartheta\sqrt{\bar{X}_{t^n_k}}\,\Delta W_{t^n_{k+1}}.$$

  - Existence (positivity !) ? still not so clear.
  - Martingality of Brownian term ? YES.

- However, still searching for a fully implicit scheme,

$$\sqrt{\bar{X}_{t^n_k}}\Delta W_{t^n_{k+1}} = \sqrt{\bar{X}_{t^n_{k+1}}}\Delta W_{t_{k+1}} - \Big(\sqrt{\bar{X}_{t^n_{k+1}}} - \sqrt{\bar{X}_{t^n_k}}\Big)\Delta W_{t_{k+1}}.$$

- Itô's Lemma applied to $(\sqrt{X_t})_t$ itself between t^n_k and t^n_{k+1} yields

$$\begin{aligned}\sqrt{X_{t^n_{k+1}}} - \sqrt{X_{t^n_k}} &= \int_{t^n_k}^{t^n_{k+1}} \big[\dots\big]dt + \int_{t^n_k}^{t^n_{k+1}} \frac{\vartheta\sqrt{X_s}}{2\sqrt{X_s}}dW_s.\\ &= \underbrace{\int_{t^n_k}^{t^n_{k+1}} \big[\dots\big]dt}_{O(T/n)} + \frac{\vartheta}{2}\Delta W_{t^n_{k+1}}.\end{aligned}$$

- Hence, conditioning by $\mathcal{F}^W_{t^n_k}$ yields,

$$\mathbb{E}\Big[\Big(\sqrt{X_{t^n_{k+1}}}-\sqrt{X_{t^n_k}}\Big)\Delta W_{t^n_{k+1}}\,\big|\mathcal{F}^W_{t^n_k}\Big] = \frac{\vartheta}{2}\mathbb{E}(\Delta W_{t^n_{k+1}})^2 + O\big((\tfrac{T}{n})^{3/2}\big)$$
$$\sim \frac{\vartheta T}{2n} = \kappa.\frac{T}{n}.\frac{\vartheta}{2\kappa}.$$

- This suggests to set $\bar{X}_0 = x$ and

$$\bar{X}_{t^n_{k+1}} = \bar{X}_{t^n_k}+\kappa\left(a-\frac{\vartheta^2}{2\kappa}-\bar{X}_{t^n_{k+1}}\right)\frac{T}{n}+\vartheta\sqrt{\bar{X}_{t^n_{k+1}}}\Delta W_{t^n_{k+1}},\; k=0:n-1.$$

Implicit scheme I

- The above equation

$$\bar{X}_{t^n_{k+1}} = \bar{X}_{t^n_k} + \kappa\left(a - \frac{\vartheta^2}{2\kappa} - \bar{X}_{t^n_{k+1}}\right)\frac{T}{n} + \vartheta\sqrt{\bar{X}_{t^n_{k+1}}}\Delta W_{t^n_{k+1}},\ k = 0 : n-1.$$

can be inverted into the straightforwardly implementable: $\bar{X}_0 = x$ and

$$\bar{X}_{t^n_{k+1}} = \left(\frac{\vartheta\Delta W_{t^n_{k+1}} + \sqrt{(\vartheta\Delta W_{t^n_{k+1}})^2 + 4\left(\kappa\left(a - \frac{\vartheta^2}{2\kappa}\right)\frac{T}{n} + \bar{X}_{t^n_k}\right)\left(1 + \frac{\kappa T}{n}\right)}}{2\left(1 + \frac{\kappa T}{n}\right)}\right)^2,$$

$k = 0, \ldots, n-1$.

- Expansions still preserving positivity can be derived from this expression.

Implicit scheme II

- Set $Y_t = \sqrt{X_t}$. Itô's Lemma shows that $(Y_t)_{t\geqslant 0}$ satisfies the *SDE*

$$(E) \quad \equiv \quad dY_t = \frac{\kappa a}{2}\left(\frac{c}{Y_t} - \frac{Y_t}{a}\right)dt + \frac{\vartheta}{2}dW_t, \quad Y_0 = \sqrt{x},$$

where $c = 1 - \frac{\vartheta^2}{4\kappa a} \geqslant \frac{\vartheta^2}{4\kappa a} > 0$.

- We associate the completely implicited Euler scheme of this equation:

$$\begin{aligned}\bar{Y}_0 &= \sqrt{x},\\ \bar{Y}_{t^n_{k+1}} &= \bar{Y}_{t^n_k} + \frac{\kappa a}{2}\left(\frac{c}{\bar{Y}_{t^n_{k+1}}} - \frac{\bar{Y}_{t^n_{k+1}}}{a}\right)\frac{T}{n} + \frac{\vartheta}{2}\Delta W_{t^n_{k+1}}.\end{aligned}$$

- Which in turn can be entirely explicited into $\bar{Y}_0 = \sqrt{x}$ and

$$\bar{Y}_{t^n_{k+1}} = \frac{\bar{Y}_{t^n_k} + \frac{\vartheta}{2}\Delta W_{t^n_{k+1}} + \sqrt{(\bar{Y}_{t^n_k} + \frac{\vartheta}{2}\Delta W_{t^n_{k+1}})^2 + 2a\kappa c\frac{T}{n}(1+\frac{\kappa T}{2n})}}{2(1+\frac{\kappa T}{2n})}.$$

- Set $\bar{X}_{t_k^n} = (\bar{Y}_{t_k^n})^2$, $k = 0 : n-1$.
- Implicit schemes have been introduced and deeply investigated by A. Alfonsi and D.Brigo (strong and weak error).
- Empirical test show that the performances of the schemes degrade as soon as

$$\frac{\vartheta^2}{2\kappa a} \approx 1.$$

although one still has $\frac{\vartheta^2}{4\kappa a} \leqslant 1$.

Milstein scheme (more about $1D$-)

- Diffusion with drift $b(t,x)$ and diffusion coefficient $\sigma(t,x)$.

  Temporarily denote the time step $h = \frac{T}{n}$.

- The Milstein operator reads

$$\mathcal{M}_h(t,x,z) = x + hb(t,x) + \sqrt{h}\sigma(t,x)z + \frac{1}{2}\sigma\sigma'_x(t,x)h(z^2-1)$$

- It can be rewritten formally as

$$\mathcal{M}(t,x,z) = \underbrace{x - \frac{\sigma(t,x)}{2\sigma'_x(t,x)}}_{(1)} + h\underbrace{\left(b(t,x) - \frac{\sigma\sigma'_x(t,x)}{2}\right)}_{(2)} + \frac{\sigma\sigma'_x(t,x)}{2}\underbrace{\left(z\sqrt{h} + \frac{1}{\sigma'_x(t,x)}\right)^2}_{(3)\quad >0\ a.e.(!)}.$$

- It can be rewritten formally as

$$\mathcal{M}(t,x,z) = \underbrace{x - \frac{\sigma(t,x)}{2\sigma'_x(t,x)}}_{(1)} + h\underbrace{\left(b(t,x) - \frac{\sigma\sigma'_x(t,x)}{2}\right)}_{(2)}$$

$$+\frac{\sigma\sigma'_x(t,x)}{2}\underbrace{\left(z\sqrt{h} + \frac{1}{\sigma'_x(t,x)}\right)^2}_{(3)\quad >0\ a.e.(!)}.$$

- If, for every $t \in [0,T]$ and every $x > 0$

$$\begin{cases} (i) & \sigma(t,x) > 0,\ \sigma'_x(t,x) > 0 \quad \text{(in fact } \sigma\sigma'_x(t,x) > 0\text{)} \\ (ii) & \dfrac{\sigma(t,x)}{2\sigma'_x(t,x)} \leqslant x \qquad \Longrightarrow (1) \geqslant 0 \\ (iii) & \dfrac{\sigma\sigma'_x(t,x)}{2} \leqslant b(t,x) \qquad \Longrightarrow (2) \geqslant 0 \end{cases}$$

whereas $(3) > 0$ $\mathbb{P}_Z(dz)$-$a.s.$ if $Z \stackrel{d}{=} \mathcal{N}(0;1)$.

- Under these assumptions, if $\bar{X}_0^{mil} = X_0 > 0$ *a.s.* then

$$\mathbb{P}\text{-}a.s. \quad \bar{X}^{mil}_{t_k^n} > 0,\ k = 0 : n$$

Exercise. This extends to the *continuous* Milstein scheme (show h plays no role).

- Application to the *CIR* process.

$$\sigma(t,x) = \vartheta\sqrt{x} > 0, \quad \sigma'_x(t,x) = \frac{\vartheta}{2\sqrt{x}} > 0 \ \text{ and } \ b(t,x) = \kappa(a-x)$$

so that (i) is satisfied and

$$\frac{\sigma(t,x)}{2\sigma'_x(t,x)} = x \Longrightarrow (ii) \text{ is satisfied as well}$$

but (iii) always fails.

Boosted *CIR* process

- Set $Y_t = e^{\kappa t} X_t$, where $(X_t)_t$ is the above *CIR*.
- $(Y_t)_t$ is called the boosted *CIR* process. By Itô formula, it satisfies the *SDE*

$$dY_t = \tilde{b}(t, Y_t)dt + \tilde{\sigma}(t, Y_t)dW_t, \quad Y_0 = x_0 > 0,$$

(Exercise!) with

$$\tilde{b}(t, y) = \kappa e^{\kappa t} a \quad \text{and} \quad \tilde{\sigma}(t, y) = e^{\frac{\kappa}{2} t} \vartheta \sqrt{y}.$$

- Then (i) and (ii) are still clearly satisfied and (iii) reads

$$\frac{\vartheta^2}{4} e^{\kappa t} \leqslant \kappa e^{\kappa t} a \qquad \Longleftrightarrow \qquad \frac{\vartheta^2}{4\kappa a} \leqslant 1 \text{ !!!}$$

Always satisfied under the strong existence condition for the (strictly) positive solution to *CIR* $\left(\frac{\vartheta^2}{2\kappa a} \leqslant 1\right)$.

- The Milstein scheme of Y is *a.s.* (strictly) positive.

ℵ Practitioner's corner

- Write the Milstein scheme for $(Y_t^x)_{t\in[0,T]}$ (which preserves its positiveness).
- Set $\widetilde{X}_{t_k^n}^x = e^{-\kappa t_k^n}\widetilde{Y}_{t_k^n}^{x,mil}$ in the simulation.
- Warning! The boosted *CIR* increases the variance of the simulation.
- A reference formula for the Laplace transform (e.g. as a reference for numerical tests):

$$\forall\,\lambda>0,\qquad \mathbb{E}\,e^{-\lambda X_T^x}=\frac{1}{(2\lambda L+1)^{\frac{2\kappa a}{\vartheta^2}}}\exp\Big(-\frac{\lambda L\zeta}{2\lambda L+1}\Big)$$

with

$$L=\frac{\vartheta^2}{4\kappa}(1-e^{-\kappa T})\quad\text{and}\quad \zeta=\frac{4x\kappa}{\vartheta^2(e^{\kappa T}-1)}.$$

- **(Simpler!) Application to the Black-Scholes model.**

$$b(x) = r\,x, \quad \sigma(x) = \sigma\,x, \quad x > 0,$$

so that $(i) \equiv \sigma(x), \sigma'(x) > 0$ is clearly satisfied. Then

$$\frac{\sigma(x)}{2\sigma'(x)} = \frac{x}{2} \leqslant x \Longrightarrow (ii) \quad \text{is satisfied}$$

and $(iii) \equiv \frac{\sigma\sigma'(x)}{2} \leqslant b(x)$, $x > 0$, reads

$$\frac{\sigma^2 x}{2} \leqslant r\,x \quad \Longleftrightarrow \quad r - \frac{\sigma^2}{2} \geqslant 0.$$

- If $r - \frac{\sigma^2}{2} < 0$, like for the *CIR*, one may introduce a "numerical" boosting factor ρ: set

$$\rho = \frac{\sigma^2}{2} - r > 0 \quad \text{and} \quad Y_t = e^{\rho t} X_t.$$

Higher dimensional Milstein scheme

- We start from the (scalar) *SDE* ($d = 1$ but $q = 2$) driven by $W = (W^1, W^2)$:

$$dX_t^x = b(X_t^x)dt + \sigma_1(X_t^x)dW_t^1 + \sigma_2(X_t^x)dW_t^2, \quad X_0^x = x.$$

- Reasoning like in 1-dimension yields shows that the 1^{st} order term $\sigma(x)\sigma'(x)\int_0^t W_s dW_s$ now reads

$$\sum_{i,j=1,2} \sigma_i'(x)\sigma_j(x)\int_0^t W_s^i dW_s^j.$$

- In particular this sum contains

$$\sigma_1'(x)\sigma_2(x)\int_0^t W_s^1 dW_s^2 + \sigma_1(x)\sigma_2'(x)\int_0^t W_s^2 dW_s^1.$$

- Unfortunately, efficient simulation methods of the triplet $\left(W_t^1, W_t^2, \int_0^t W_s^1 dW_s^2\right)$ (which would be enough) remains out of reach ...

- Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = [\sigma_{ij}] : \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R})$ (differentiable) and W a q-dimensional B.M.
- The (d, q)-dimensional Milstein scheme reads

Milstein scheme

$$
\begin{aligned}
\bar{X}_0^{mil} &= X_0,\\
\bar{X}_{t_{k+1}^n}^{mil} &= \bar{X}_{t_k^n}^{mil} + \frac{T}{n} b(\bar{X}_{t_k^n}^{mil}) + \sigma(\bar{X}_{t_k^n}^{mil})\Delta W_{t_{k+1}^n}\\
&\quad + \sum_{1\leqslant i,j\leqslant q} \partial\sigma_{.i}\sigma_{.j}(\bar{X}_{t_k^n}^{mil}) \times \int_{t_k^n}^{t_{k+1}^n} (W_s^i - W_{t_k^n}^i)dW_s^j, \qquad k = 0 : n-1
\end{aligned}
$$

where, for every $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, the tensors $\partial\sigma_{.i}\,\sigma_{.j}$ read

$$
\boxed{\partial\sigma_{.i}\,\sigma_{.j}(x) := \sum_{\ell=1}^{d} \frac{\partial\sigma_{.i}}{\partial x^\ell}(x)\sigma_{\ell j}(x) \in \mathbb{R}^d \text{ with } \sigma = [\sigma_{.i}]_{i=1,\ldots,q}.}
$$

Proposition [Commuting case]

If the tensor $\partial\sigma_{.i}\,\sigma_{.j}$ commute, *i.e.*

$$\forall\, i \neq j, \qquad \partial\sigma_{.i}\,\sigma_{.j} = \partial\sigma_{.j}\,\sigma_{.i},$$

then the Milstein scheme can be simulated since it reduces to

$$\begin{aligned}\bar X^{mil}_{t^n_{k+1}} &= \bar X^{mil}_{t^n_k} + \frac{T}{n}\Big(b(\bar X^{mil}_{t^n_k}) - \frac12\sum_{i=1}^{q}\partial\sigma_{.i}\sigma_{.i}(\bar X^{mil}_{t^n_k})\Big) + \sigma(\bar X^{mil}_{t^n_k})\Delta W_{t^n_{k+1}}\\ &\quad + \frac12\sum_{1\leqslant i,j\leqslant q}\partial\sigma_{.i}\,\sigma_{.j}(\bar X^{mil}_{t^n_k})\Delta W^i_{t^n_{k+1}}\Delta W^j_{t^n_{k+1}}, \quad \widetilde X^{mil}_0 = X_0.\end{aligned}$$

Proof. Let $i \neq j$. As $\partial\sigma_{.i}\,\sigma_{.j} = \partial\sigma_{.j}\,\sigma_{.i}$, Lévy areas related to (W^i, W^j) only appear through their sum. Now, an integration by parts yields

$$\int_{t^n_k}^{t^n_{k+1}}(W^i_s - W^i_{t^n_k})dW^j_s + \int_{t^n_k}^{t^n_{k+1}}(W^j_s - W^j_{t^n_k})dW^i_s = \Delta W^i_{t^n_{k+1}}\Delta W^j_{t^n_{k+1}}$$

since W^i and W^j are independent.

- **Example:** *Multi-dimensional Black-Scholes model*.

$$dX_t^i = X_t^i\Big(r dt + \sum_{j=1}^{q} \sigma_{ij} dW_t^j\Big),\; X_0^i = x_0^i > 0,\; i = 1 : d,$$

where $W = (W^1, \ldots, W^q)$ a q-dimensional B.M.

Then, for every $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$,

$$\forall\, i, j \in \{1, \ldots, q\}, \quad \partial\sigma_{.i}\,\sigma_{.j}(x) = \Big[\sigma_{ki}\sigma_{kj}x^k\Big]_{k=1:d} \in \mathbb{R}^d$$

so that, the Black-Scholes tensors do commute.

- **Exercise**. Prove it!

- A theoretical result:

Theorem (Multi-dimensional Milstein scheme)

Assume that b and σ are $\mathcal{C}^1$ on $\mathbb{R}^d$ with bounded α'_b and α'_σ-Hölder continuous existing partial derivatives respectively. Let $\alpha = \min(\alpha'_b, \alpha'_\sigma)$. Then, for every $p \in (0, +\infty)$, there exists a real constant $C_{p,b,\sigma,T} > 0$ such that for every $X_0 \in L^p(\mathbb{P})$, independent of the q-dimensional Brownian motion W, the error bound established when $d = q = 1$ remains valid i.e.

$$\Big\| \max_{0 \leqslant k \leqslant n} \big| X_{t_k^n} - \bar{X}^{mil,n}_{t_k^n} \big| \Big\|_p \leqslant \Big\| \sup_{t \in [0,T]} \big| X_t - \bar{X}^{mil,n}_t \big| \Big\|_p \leqslant C_{b,\sigma,T,p} \left(\frac{T}{n} \right)^{\frac{1+\alpha}{2}} (1 + \|X_0\|_p)$$

where $\alpha = \min(\alpha_{b'}, \alpha_{\sigma'})$.

- A major drawback: In general the continuous Milstein scheme cannot be simulated.
- Mostly useful at time T.