Warning (what should never be done)!

- If using two *independent* samples $(Z_k)_{k\geqslant 1}$ and $(\widetilde{Z}_k)_{k\geqslant 1}$ of the distribution of Z to simulate copies $F(x-\varepsilon, Z)$ and $F(x+\varepsilon, Z)$.
- Then,

$$
\begin{aligned}
\mathrm{Var}\Big(\frac{1}{M}\sum_{k=1}^{M}\frac{F(x+\varepsilon, Z_k)-F(x-\varepsilon,\widetilde{Z}_k)}{2\varepsilon}\Big)
&= \frac{1}{4M\varepsilon^2}\Big(\mathrm{Var}\big(F(x+\varepsilon, Z)\big)+\mathrm{Var}\big(F(x-\varepsilon, Z)\big)\Big)\\
&\simeq \frac{\mathrm{Var}\big(F(x, Z)\big)}{2M\varepsilon^2}.
\end{aligned}
$$

- The asymptotic variance of the estimator explodes as $\varepsilon \to 0$ so that

$$
RMSE \simeq [f'']_{\mathrm{Lip}}\frac{\varepsilon^2}{2}+\frac{\mathrm{Std}\big(F(x, Z)\big)}{\varepsilon\sqrt{2M}},
$$

- Leads to unrealistic constraint $M(\varepsilon) \propto \varepsilon^{-6}$ to keep the balance between the bias and variance term (one hundred times more simulations!).
- Equivalently to switch $\varepsilon \rightsquigarrow \varepsilon/\sqrt{2}$ and $M \rightsquigarrow 8\,M$ to reduce the error by a 2-factor.

Examples (Greeks computation) I.

- *Sensitivity in a Black–Scholes model.*
  - Vanilla payoffs $h : \mathbb{R}_+ \to \mathbb{R}$ viewed as functions of $\mathcal{N}(0;1)$ distribution

$$F(x,z) = e^{-rT} h\left(x\, e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}\right),\ z \in \mathbb{R},\ x \in (0,+\infty),$$

  - If h is Lipschitz continuous, then

$$|F(x',Z) - F(x'',Z)| \leqslant [h]_{\rm Lip}|x'-x''|e^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}Z}$$

  - Elementary computations show, using that $Z \stackrel{d}{=} \mathcal{N}(0;1)$,

$$\forall\, x',\, x'' \in \mathbb{R},\quad \|F(x',Z) - F(x'',Z)\|_2 \leqslant [h]_{\rm Lip}|x'-x''|e^{\frac{\sigma^2}{2}T}.$$

  - Regularity of f follows from an easy change of variable ($\mu = r - \frac{\sigma^2}{2}$)

$$f(x) = e^{-rT}\int_{\mathbb{R}} h\Big(x\, e^{\mu T+\sigma\sqrt{T}z}\Big)e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} = e^{-rT}\int_0^{+\infty} h(y)e^{-\frac{(\log(x/y)+\mu T)^2}{2\sigma^2 T}}\frac{dy}{y\sigma\sqrt{2\pi T}}$$

    The second integral appears as a $\frac{dy}{y}$-convolution on $(0,+\infty)$.
  - Hence f is infinitely differentiable on $(0,+\infty)$.
  - So that $f^{(k)}$ are loc-Lipschitz Lipschitz on $(0,+\infty)$.

Examples (Greeks computation) II.

- *Diffusion model with Lipschitz continuous coefficients.* Let $X^x = (X_t^x)_{t\in[0,T]}$ denote the Brownian diffusion solution of

$$(SDE) \;\equiv\; dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x$$

  where b and σ are Lipschitz.
- We should instead write $F(x,\omega) = h\big(X_T^x(\omega)\big)$.
- Lipschitz continuity of the flow of the above SDE yields

$$\forall\, x,\, x' \in \mathbb{R}^d, \quad \big\|F(x,\,.\,) - F(x',\,.\,)\big\|_2 \leqslant C_{b,\sigma}[h]_{\rm Lip}|x - x'|e^{C_{b,\sigma}T}$$

  where $C_{b,\sigma}$ is a positive constant only depending on the Lipschitz coefficients of b and σ.
- Also holds for $\|\cdot\|_{\sup}$-Lip path-dependent functionals.
- Regularity of $f(x) = \mathbb{E}\,F(x,\omega)$ is less straightforward...
- Holds in two situations: $\triangleright$ h, b and σ are regular enough to apply pathwise differentiation of $x \mapsto F(x,\omega)$
  $\triangleright$ or σ satisfies a uniform ellipticity assumption $\sigma \geqslant \varepsilon_0 > 0$.

Examples (Greeks computation) III. Euler scheme, path-dependence.

- *Euler scheme of a Brownian diffusion model with Lipschitz continuous coefficients.* The same holds for the Euler scheme. Furthermore, Assumption (128) holds uniformly with respect to n if $\frac{T}{n}$ is the step size of the Euler scheme.
- F can also be a functional of the whole path of a diffusion, provided F is Lipschitz continuous with respect to the sup-norm over $[0, T]$ i.e. is $\|\cdot\|_{\sup}$-Lipschitz functional from $C([0,T],\mathbb{R}^d) \to \mathbb{R}$.

Singular setting

- Above setting often testifies of laziness (no excuse now with *AAD*. . .)
- But in the case of *singular setting*: f is differentiable at x but $F(\,.\,,Z)$ is not L^2-Lipschitz.

Proposition (Singular setting)

Let $x \in \mathbb{R}$.

- Assume that F satisfies an $L^2(\mathbb{P})$ θ-Hölder assumption, $\theta \in (0,1]$ in the neighbourhood of x: there exists a positive real constant $C_{Hol,F,Z}$

$$\forall\, x', x'' \in (x-\varepsilon_0, x+\varepsilon_0), \quad \big\|F(x',Z)-F(x'',Z)\big\|_2 \leqslant C_{Hol,F,Z}|x'-x''|^{\theta}.$$

- f is twice differentiable with f'' Lipschitz on $(x-\varepsilon_0, x+\varepsilon_0)$.
- Let $(Z_k)_{k\geqslant 1}$ be a sequence of i.i.d. random vectors with the same distribution as Z.
- Then, for every $\varepsilon \in (0,\varepsilon_0)$, the *RMSE* satisfies

$$\left\| f'(x) - \frac{1}{M}\sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon} \right\|_2 \leqslant \sqrt{\left([f'']_{\rm Lip}\frac{\varepsilon^2}{2}\right)^2 + \frac{C^2_{Hol,F,Z}}{(2\varepsilon)^{2(1-\theta)}M}}$$

$$\leqslant [f'']_{\rm Lip}\frac{\varepsilon^2}{2} + \frac{C_{Hol,F,Z}}{(2\varepsilon)^{1-\theta}\sqrt{M}}.$$

ℵ Practitioner's corner

- To divide the error by a 2-factor

$$\varepsilon \rightsquigarrow \varepsilon/\sqrt{2} \quad \text{and} \quad M \rightsquigarrow 2^{3-\theta} M = 2^{1-\theta} \cdot 4M$$

- **Exercise.** Prove the above Proposition.

Example, part I

- *Pricing digital option/hedging a Call option in a B-S model.*
  - Let
$$h(\xi) = \mathbf{1}_{\{\xi \geqslant K\}}$$
  - Set $\mu = r - \frac{\sigma^2}{2}$ and
$$F(x,z) = e^{-rT} h\left(x\, e^{\mu T + \sigma\sqrt{T}z}\right),\ z \in \mathbb{R},\ x \in (0,+\infty)$$
  - Premium given by
$$f(x) = \mathbb{E}\, F(x,Z) = e^{-rT}\mathbb{P}\Big(x\, e^{\mu T+\sigma\sqrt{T}Z} \geqslant K\Big) \quad \text{with} \quad Z \stackrel{d}{=} \mathcal{N}(0;1).$$
  - As $Z \stackrel{d}{=} -Z$ (with Φ_0 the c.d.f. $\mathcal{N}(0;1)$) we get
$$\begin{aligned} f(x) &= e^{-rT}\mathbb{P}\Big(x\, e^{\mu\, T+\sigma\sqrt{T}Z} \geqslant K\Big) \\ &= e^{-rT}\mathbb{P}\Big(Z \geqslant -\frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\Big) = e^{-rT}\Phi_0\Big(\frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\Big). \end{aligned}$$
  - Hence the function f is infinitely differentiable on $(0,+\infty)$.

Example, part II

- Still using $Z \stackrel{d}{=} -Z$, for every $x, x' \in \mathbb{R}$,

$$
\begin{aligned}
\|F(x,Z)-F(x',Z)\|_2^2 &= e^{-2rT}\left\|\mathbf{1}_{\left\{Z\geqslant -\frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\right\}}-\mathbf{1}_{\left\{Z\geqslant -\frac{\log(x'/K)+\mu T}{\sigma\sqrt{T}}\right\}}\right\|_2^2\\
&= e^{-2rT}\mathbb{E}\left|\mathbf{1}_{\left\{Z\leqslant \frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\right\}}-\mathbf{1}_{\left\{Z\leqslant \frac{\log(x'/K)+\mu T}{\sigma\sqrt{T}}\right\}}\right|^2\\
&= e^{-2rT}\left(\Phi_0\Big(\frac{\log(\max(x,x')/K)+\mu T}{\sigma\sqrt{T}}\Big)-\Phi_0\Big(\frac{\log(\min(x,x')/K)+\mu T}{\sigma\sqrt{T}}\Big)\right).
\end{aligned}
$$

- Using that $\Phi_0'(\xi)=\kappa_0 e^{-\xi^2/2}$ is bounded by $\kappa_0=\frac{1}{\sqrt{2\pi}}$, we derive that

$$
\big\|F(x,Z)-F(x',Z)\big\|_2^2 \leqslant \frac{\kappa_0 e^{-2rT}}{\sigma\sqrt{T}}\left|\log x-\log x'\right|.
$$

- Consequently for every interval $I\subset(0,+\infty)$ *bounded away from* 0, there exists a real constant $C_{r,\sigma,T,I}>0$ such that

$$
\forall\, x,\, x'\in I,\quad \|F(x,Z)-F(x',Z)\|_2 \leqslant C_{r,\sigma,T,I}\sqrt{|x-x'|},
$$

i.e. the functional F is $\frac{1}{2}$-Hölder in $L^2(\mathbb{P})$.

Decreasing step shock method

- To get rid of the bias we will introduce a decreasing step sequence $(\varepsilon_k)_{k\geqslant 1}$ of positive real numbers decreasing to 0.
- Consider the estimator

$$\widehat{f'(x)}_M := \frac{1}{M}\sum_{k=1}^{M} \frac{F(x+\varepsilon_k, Z_k) - F(x-\varepsilon_k, Z_k)}{2\varepsilon_k}.$$

- It can be computed in a recursive way since

$$\widehat{f'(x)}_{M+1} = \widehat{f'(x)}_M + \frac{1}{M+1}\left(\frac{F(x+\varepsilon_{M+1}, Z_{M+1}) - F(x-\varepsilon_{M+1}, Z_{M+1})}{2\varepsilon_{M+1}} - \widehat{f'(x)}_M\right).$$

- Elementary computations show that the mean squared error satisfies

$$
\begin{aligned}
\Big\| f'(x) - \widehat{f'(x)}_M \Big\|_2^2 &= \left(f'(x) - \frac{1}{M}\sum_{k=1}^{M} \frac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{2\varepsilon_k} \right)^2 \\
&\quad + \frac{1}{M^2}\sum_{k=1}^{M} \frac{\mathrm{Var}(F(x+\varepsilon_k, Z_k) - F(x-\varepsilon_k, Z_k))}{4\varepsilon_k^2} \\
&\leqslant \frac{[f'']_{\mathrm{Lip}}^2}{4M^2}\left(\sum_{k=1}^{M}\varepsilon_k^2\right)^2 + \frac{C_{F,Z}^2}{M} \\
&= \frac{1}{M}\left(\frac{[f'']_{\mathrm{Lip}}^2}{4M}\left(\sum_{k=1}^{M}\varepsilon_k^2\right)^2 + C_{F,Z}^2 \right)
\end{aligned}
$$

- As a consequence, the *RMSE* satisfies

$$
\Big\| f'(x) - \widehat{f'(x)}_M \Big\|_2 \leqslant \frac{1}{\sqrt{M}} \sqrt{\frac{[f'']_{\mathrm{Lip}}^2}{4M}\left(\sum_{k=1}^{M}\varepsilon_k^2\right)^2 + C_{F,Z}^2}.
$$

L^2-rate of convergence (erasing the asymptotic bias)

- How to get a $\frac{1}{\sqrt{M}}$ L^2-rate like in a standard MC simulation ? Erase the bias!
- The bias term will fade as $M \to +\infty$ if

$$\frac{1}{M}\left(\sum_{k=1}^{M} \varepsilon_k^2\right)^2 \to 0 \quad \text{as} \quad M \to +\infty.$$

- This leads us to choose ε_k of the for $\varepsilon_k = o\big(k^{-\frac{1}{4}}\big)$ since, then, $\displaystyle\sum_{k=1}^{M} \varepsilon_k^2 = o\left(\sum_{k=1}^{M} \frac{1}{k^{\frac{1}{2}}}\right)$ as $M \to +\infty$ so that

$$\sum_{k=1}^{M} \frac{1}{k^{\frac{1}{2}}} = \sqrt{M}\,\frac{1}{M}\sum_{k=1}^{M}\left(\frac{k}{M}\right)^{-\frac{1}{2}} \sim \sqrt{M}\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{M} \;\text{ as }\; M \to +\infty.$$

- However, the choice of too small steps ε_k may introduce numerical instability in the computations, so we recommend to choose the ε_k of the form

$$\varepsilon_k = o\big(k^{-(\frac{1}{4}+\delta)}\big) \quad \text{with small enough} \quad \delta > 0.$$

- One can slightly refine the above bound obtained:

$$\sum_{k=1}^{M} \frac{\mathrm{Var}(F(x+\varepsilon_k, Z_k) - F(x-\varepsilon_k, Z_k))}{4\varepsilon_k^2} = \sum_{k=1}^{M} \frac{\big\|F(x+\varepsilon_k, Z_k) - F(x-\varepsilon_k, Z_k)\big\|_2^2}{2\varepsilon_k}$$
$$-\sum_{k=1}^{M} \left(\frac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{2\varepsilon_k}\right)^2.$$

- Now, since $\dfrac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{2\varepsilon_k} \to f'(x)$ as $k \to +\infty$,

$$\frac{1}{M}\sum_{k=1}^{M} \left(\frac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{4\varepsilon_k^2}\right)^2 \to f'(x)^2 \quad \text{as} \quad M \to +\infty.$$

- Plugging this into the above computations yields the refined asymptotic upper-bound

$$\overline{\lim_{M\to+\infty}} \sqrt{M} \left\| f'(x) - \widehat{f'(x)}_M \right\|_2 \leqslant \sqrt{C_{F,Z}^2 - \big(f'(x)\big)^2}.$$

- This approach has the same quadratic rate of convergence as a regular MC simulation (*e.g.* a simulation carried out with $\frac{\partial F}{\partial x}(x,Z)$ if it exists).

Proposition (*A.s.* convergence of the estimator)

Under the assumptions on F and f made in the constant step setting and if ε_k goes to zero as k goes to infinity, the estimator

$$\widehat{f'(x)}_M \longrightarrow f'(x) \quad \mathbb{P}\text{-}a.s. \quad \text{as} \quad M \to +\infty.$$

- Ingredients of proof: discrete time martingale convergence theorems (cf. poly/book).

Tangent process and pathwise differentiability

- Back to the standard *SDE* starting from $X_0 = x \in \mathbb{R}^d$:

$$(SDE) \;\equiv\; X_t = x + \int_0^t b(s, X_s)ds + \sigma(s, X_s)dW_s.$$

- b and σ continuous on $[0, T] \times \mathbb{R}^d$
- $b(t,\cdot),\ \sigma(t,\cdot) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ i.e. $b'_{x^i}(t,\cdot)$, $\sigma'_{x^i}(t,\cdot)$ α-Hölder in x, U in t.

Theorem (Kunita's flow theorem)

(a) *At every* $t \in \mathbb{R}_+$, *the mapping* $x \mapsto X_t^x$ *is a.s. continuously differentiable and its Jacobian* $Y_t(x) := \nabla_x X_t^x = \left[\dfrac{\partial (X_t^x)^i}{\partial x^j}\right]_{1\leqslant i,j\leqslant d}$ *satisfies the linear stochastic differential system:*

$$Y_t^{ij}(x) = \delta_{ij} + \sum_{\ell=1}^{d}\int_0^t \frac{\partial b^i}{\partial y^\ell}(s, X_s^x)Y_s^{\ell j}(x)ds + \sum_{\ell=1}^{d}\sum_{k=1}^{q}\int_0^t \frac{\partial \sigma_{ik}}{\partial y^\ell}(s, X_s^x)Y_s^{\ell j}(x)dW_s^k,$$

$1 \leqslant i,j \leqslant d$, *where* δ_{ij} *denotes the Kronecker symbol.*

Theorem

(b) *Furthermore, the* tangent process $Y^{(x)}$ *takes values in the set* $GL(d,\mathbb{R})$ *of invertible square matrices so that, for every* $t \geqslant 0$, *the mapping* $x \mapsto X_t^x$ *is a.s. a homeomorphism of* $\mathbb{R}^d$.

- When $d = q = 1$, $Y_t^{(x)}$ satisfies

$$dY_t^{(x)} = b'_x(t, X_t^x)Y_t^{(x)}dt + \sigma'_x(t, X_t^x)Y_t^{(x)}dW_t, \quad Y_0^{(x)}(x) = 1$$

- so that

$$Y_t^{(x)} = \exp\Big(\int_0^t \big[b'_x(s, X_s^x) - \tfrac{1}{2}\sigma'_x(s, X_s^x)^2\big]ds + \int_0^t \sigma'_x(s, X_s^x)dW_s\Big)$$

Applications to δ-hedging

- Tangent process and the δ-hedge are closely related.
- Set (assume $r = 0$)
$$f(x) := \mathbb{E}\, h\big(X_T^x\big).$$
- The δ-hedge vector of this option (at time 0 and) at $x = (x^1, \ldots, x^d) \in (0, +\infty)^d$ is given by $\nabla f(x)$.
- Representation of $\nabla f(x)$ as an expectation (with in view its computation by a Monte Carlo simulation).

Proposition (δ-hedge of vanilla European options)

If $h : \mathbb{R}^d \to \mathbb{R}$ is Borel and satisfies :

(i) *A.s.* differentiability: $\nabla h(y)$ *exists* $\mathbb{P}_{X_T^x}(dy)$*-a.s.,*

(ii) Uniform Integrability condition (UI)*:*

$$\left(\frac{|h(X_T^x) - h(X_T^{x'})|}{|x - x'|}\right)_{|x'-x|<\varepsilon_0,\, x'\neq x} \qquad \textit{is U.I..}$$

Proposition (sequel)

Then, f is differentiable at x and $\nabla f(x)$ reads

$$\nabla f(x) = \mathbb{E}\left[(Y_T^{(x)})^* \nabla h(X_T^x)\right], \; i = 1, \ldots, d.$$

- If h is Lipschitz then

$$\left|h(X_T^x) - h(X_T^{x'})\right| \leqslant [h]_{\rm Lip}|X_T^x - X_T^{x'}|$$

and $\left(\frac{|X_T^x - X_T^{x'}|}{|x-x'|}\right)_{|x'-x|<\varepsilon_0, x'\neq x}$ is $U.I.$ since $L^{1+\eta}(\mathbb{P})$-bounded [admitted]

- Other parameters: Let $b(\theta, t, x)$ and $\sigma(\theta, t, x)$. How to proceed?
- Trick: add a component to X

$$d\theta_t = 0 \cdot dt + 0 \cdot dW_t, \quad \theta_0 = \theta$$
$$dX_t = b(\theta, t, x_t^X)dt + \sigma(\theta, t, X_t^x)dW_t, \quad X_0 = x.$$

Non-smooth payoffs: log-likelihood method

- We know that if

$$X(\theta) \stackrel{d}{=} \underbrace{p(\theta, y)}_{>0\,\mu(dy)\,a.e.\forall\,\theta} \quad \mu(dy), \quad \theta \in \Theta \subset \mathbb{R},$$

- then, under appropriate mathematical conditions, the function

$$f(\theta) = \mathbb{E}\, h\big(X(\theta)\big)$$

is differentiable with

$$f'(\theta) = \mathbb{E}\Big[h\big(X(\theta)\big)\partial_\theta\big(\log p(\theta, X(\theta))\big)\Big].$$

- Works in a B–S but, in general $p(\theta, y)$ is unknown.
- When $X(\theta) = X_T(\theta)$ (diffusion framework) iconoclast suggestion

$$f(\theta) = \mathbb{E}\, h\big(X_T(\theta)\big) \simeq \bar{f}(\theta) = \mathbb{E}\, h\big(\bar{X}^n_T(\theta)\big)$$

- so that (hum...)

$$f'(\theta) \simeq \bar{f}'(\theta).$$

- But let's go (in fact no choice!).

Back to diffusions and Euler scheme

- Hypothesis on the diffusion coefficient $\sigma(\theta, \xi)$

$$\frac{1}{\varepsilon_0}|\xi|^2 \leqslant \sigma\sigma^*(\theta, \xi) \leqslant \varepsilon_0|\xi|^2.$$

- Hence $q \geqslant d$.
- Reminder: $t_k^n = \frac{kT}{n}$, $k = 0, \ldots, n$.

Density of the Euler scheme

Let $q \geqslant d$. Assume $\sigma\sigma^*(\theta, x) \in GL(d, \mathbb{R})$ for every $x \in \mathbb{R}^d$ and $\theta \in \Theta$.

Proposition (Density of the Euler scheme)

(a) *Then the distribution* $\mathbb{P}_{\bar{X}^{x,n}_{\frac{T}{n}}}(dy)$ *of* $\bar{X}^{x,n}_{\frac{T}{n}}$ *is absolutely continuous with respect to the Lebesgue measure with a probability density given by*

$$\bar{p}^n_{\frac{T}{n}}(\theta, x, y) = \frac{e^{-\frac{n}{2T}\left(y-x-\frac{T}{n}b(\theta,x)\right)^*(\sigma\sigma^*(\theta,x))^{-1}\left(y-x-\frac{T}{n}b(\theta,x)\right)}}{(2\pi\, T/n)^{\frac{d}{2}}\sqrt{\det \sigma\sigma^*(\theta, x)}}.$$

(b) *The distribution* $\mathbb{P}_{(\bar{X}^{x,n}_{t^n_1},\ldots,\bar{X}^{x,n}_{t^n_k},\ldots,\bar{X}^{x,n}_{t^n_n})}(dy_1, \ldots, dy_n)$ *of the n-tuple* $(\bar{X}^{x,n}_{t^n_1}, \ldots, \bar{X}^{x,n}_{t^n_k}, \ldots, \bar{X}^{x,n}_{t^n_n})$ *has a probability density given by*

$$\bar{p}^n_{t^n_1,\ldots,t^n_n}(\theta, x, y_1, \ldots, y_n) = \prod_{k=1}^{n} \bar{p}^n_{\frac{T}{n}}(\theta, y_{k-1}, y_k) \qquad \textit{(convention: } y_0 = x\textit{)}.$$

Proof

- (a) The one step Euler scheme with step T/n starting from x reads

$$\bar{X}^{n,x}_{\frac{T}{n}} \stackrel{d}{=} \mathcal{N}\left(x + \tfrac{T}{n} b(\theta, x); \sigma\sigma^*(\theta, x)\right)$$

and the formula for the density of $\mathcal{N}(m; \Sigma)$.

- (b) by induction based on the Markov property satisfied by the Euler scheme (we drop ^{x} for simplicity):

$$\mathcal{L}\Big(\bar{X}^n_{t^n_{k+1}}(\theta)\,|\mathcal{F}^W_{t^n_k}\Big) = \mathcal{L}\Big(\bar{X}^n_{t^n_{k+1}}(\theta)\,|\bar{X}^n_{t^n_k}\Big), \quad k = 0, \ldots, n-1$$

and that

$$\mathcal{L}\Big(\bar{X}^n_{t^n_{k+1}}(\theta)\,|\bar{X}^n_{t^n_k}(\theta) = y_k\Big) = \mathcal{L}\Big(\bar{X}^n_{t^n_1}(\theta)\,|\bar{X}^n_0(\theta) = y_k\Big) = \bar{p}^n_{\frac{T}{n}}(\theta, x, y_k)\lambda_d(dx).$$

One concludes by an easy induction that, for every bounded Borel function $g : (\mathbb{R}^d)^k \to \mathbb{R}_+$,

$$\mathbb{E}\, g\big(\bar{X}^n_{t^n_1}, \ldots, \bar{X}^n_{t^n_k}\big) = \int_{(\mathbb{R}^d)^k} g(y_1, \ldots, y_k) \prod_{\ell=1}^{k} \bar{p}^n_{\frac{T}{n}}(\theta, y_{\ell-1}, y_\ell)$$

with the convention $y_0 = 0$. $\square$

ℵ Practitioner's corner

We derive the formula for the partial derivative:

$$\bar{f}'(\theta) \simeq \mathbb{E}\left(\varphi(\bar{X}^n_T(\theta))\frac{\partial \log \bar{p}^n_{t^n_1,\ldots,t^n_n}}{\partial\theta}(\theta, x, \bar{X}^x_{t^n_1}(\theta),\ldots,\bar{X}^x_{t^n_n}(\theta))\right)$$

$$\bar{f}'(\theta) = \sum_{k=1}^{n}\mathbb{E}\left(\varphi\big(\bar{X}^n_T(\theta)\big)\frac{\partial \log \bar{p}^n_{\frac{T}{n}}}{\partial\theta}\big(\theta, \bar{X}^n_{t^n_{k-1}}(\theta), \bar{X}^n_{t^n_k}(\theta)\big)\right)$$

simulable by a *path-dependent* Monte Carlo simulation.

Toward differentiation on the Wiener space: Bismut-Elworthy-Li Theorem

Theorem (BEL's formula, 1980, $d = q = 1$ here)

- *Let $X^x = (X^x_t)_{t\in[0,T]}$ be a diffusion process, unique $(\mathcal{F}^W_t)_{t\in[0,T]}$-adapted solution to*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

where b, $\sigma \in \mathcal{C}^1_b$ (hence Lipschitz). Let $\varphi \in \mathcal{C}^1_b(\mathbb{R},\mathbb{R})$ be such that

$$\mathbb{E}\Big(\varphi(X^x_T)^2 + \big(\varphi'(X^x_T)\big)^2\Big) < +\infty.$$

- *Let $(H_t)_{t\in[0,T]}$ be $\mathcal{F}^W$-progressively measurable s.t. $\mathbb{E}\int_0^T H_s^2 ds < +\infty$.*
- *Then* $\mathbb{E}\left(\varphi(X^x_T)\displaystyle\int_0^T H_s dW_s\right) = \mathbb{E}\left(\varphi'(X^x_T)Y^{(x)}_T\displaystyle\int_0^T \frac{\sigma(X^x_s)H_s}{Y^{(x)}_s}ds\right)$

where $Y^{(x)}_t = \frac{dX^x_t}{dx}$ is the tangent process of X^x at x.

Application to δ-hedge I

- If φ is smooth, say $\mathcal{C}_b^1$,

$$\partial_x \mathbb{E}\,\varphi(X_T^x) = \mathbb{E}\left[\varphi'(X_T^x)Y_T^{(x)}\right].$$

- Plugging $H_t = \dfrac{Y_t^{(x)}}{\sigma(X_t^x)}$ (which satisfies $\mathbb{E}\int_0^T H_s^2 ds < +\infty$ under ellipticity assumption) into *BEL* formula

$$\mathbb{E}\left(\varphi(X_T^x)\int_0^T H_s dW_s\right) = \mathbb{E}\left(\varphi'(X_T^x)Y_T\int_0^T \frac{\sigma(X_s^x)H_s}{Y_s}ds\right)$$

yields

$$\mathbb{E}\left[\varphi(X_T^x)\int_0^T \frac{Y_s^{(x)}}{\sigma(X_s^x)}dW_s\right] = \mathbb{E}\left[\varphi'(X_T^x)Y_T^{(x)}T\right] = T\cdot\partial_x\mathbb{E}\,\varphi(X_T^x)$$

Application to δ-hedge II

$$\boxed{\partial_x \mathbb{E}\,\varphi(X_T^x) = \mathbb{E}\left[\varphi(X_T^x)\frac{1}{T}\int_0^T \frac{Y_s^{(x)}}{\sigma(X_s^x)}dW_s\right]}$$

Exercise. Set $b(x) = r\,x$ and $\sigma(x) = \sigma x$, $\sigma > 0$ (B–S model). Check that we retrieve the "weighted" δ-hedge formula already established for this specific model.