An Introduction to Hawkes Processes with Applications to Finance

Sophie LARUELLE

LAMA-UPEC

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Outline

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Simple Point Processes

Definition. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be some probability space. Let $(\tau_i)_{i\geq 1}$ be a sequence of non-negative random variables such that $\forall i \geq 1$, $\tau_i < \tau_{i+1}$. We call $(\tau_i)_{i\geq 1}$ a (simple) point process on $\mathbb{R}_+$.

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We start counting events with index 1. If needed, we will assume that $\tau_0 = 0$.

Counting Processes and Duration

Definition (Counting process). Let $(\tau_i)_{i\geq 1}$ be a point process. The right-continuous process

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Definition (Duration). The process $(\Delta\tau_i)_{i\geq 1}$ defined by

$$\forall i \geq 1, \quad \Delta\tau_i = \tau_i - \tau_{i-1}$$

is called the duration process associated with $(\tau_i)_{i\geq 1}$.

Representation of a simple point process

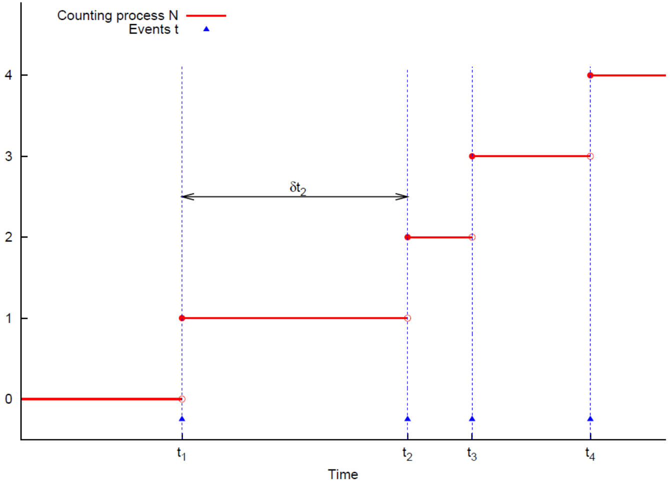

Figure: Illustration of a simple point process: events, counting process and duration process.

Intensity Process

Definition. Let N be a counting process adapted to a filtration $\mathcal{F}_t$. The left-continuous intensity process is defined as

$$\lambda(t|\mathcal{F}_t) = \lim_{h\downarrow 0} \mathbb{E}\left[\frac{N_{t+h} - N_t}{h} \mid \mathcal{F}_t\right],$$

or equivalently

$$\lambda(t|\mathcal{F}_t) = \lim_{h\downarrow 0} \frac{1}{h}\mathbb{P}\left(N_{t+h} - N_t > 0|\mathcal{F}_t\right).$$

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Intensity depends on the choice of filtration, but we will always assume that the filtration used is the natural one for the process N, denoted $\mathcal{F}_t^N$. We will therefore write $\lambda(t)$ instead of $\lambda(t|\mathcal{F}_t^N)$.

Example: Homogeneous Poisson Process

Let $\lambda \in \mathbb{R}_+^*$. A Poisson process with constant rate λ is a point process defined by

$$\begin{aligned}
\mathbb{P}(N_{t+h} - N_t = 1 | \mathcal{F}_t) &= \lambda h + o(h), \quad h \to 0^+, \\
\mathbb{P}(N_{t+h} - N_t > 1 | \mathcal{F}_t) &= o(h), \quad h \to 0^+.
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- Durations $(\Delta\tau_i)_{i\geq 1}$ of a homogeneous Poisson process are independent and identically distributed (i.i.d.) according to an exponential distribution with parameter λ.
- This process is Markovian and very often used in queueing theory (when the consumption and the service are assumed to be independent).

Stochastic Time Change

Definition. The integrated intensity function Λ is defined as

$$\forall i \geq 1, \quad \Lambda(\tau_{i-1}, \tau_i) = \int_{\tau_{i-1}}^{\tau_i} \lambda(s) ds.$$

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The following theorem illustrates the fact that any point process comes down to a Poisson process up to a time change. This result is the equivalent of the Dubins-Schwarz theorem for martingales, namely every continuous local $(\mathcal{F}_t)$-martingale M is of the form $M_t = B_{\langle M \rangle_t}$, $t \geq 0$, where $(B_s)_{s \geq 0}$ is a $\mathcal{F}_{\tau_s}$-Brownian motion (with $\tau_s = \inf\{t : \langle M \rangle_t > s\}$).

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Theorem (Time change theorem). Let N be a counting process on $\mathbb{R}_+$ such that $\int_0^\infty \lambda(s)ds = 1$. Let t_τ be the stopping time defined by

$$\int_0^{t_\tau} \lambda(s)ds = \tau.$$

Then the process $\tilde{N}_\tau = N_{t_\tau}$ is a homogeneous Poisson process with constant intensity $\lambda = 1$.

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Definition. A general definition for a linear self-exciting process N reads

$$\lambda(t) = \lambda_0(t) + \int_{-\infty}^{t} \varphi(t-s)dN_s = \lambda_0(t) + \sum_{\tau_i < t} \varphi(t-\tau_i),$$

where $\lambda_0 : \mathbb{R} \mapsto \mathbb{R}_+$ is a deterministic base intensity and $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ expresses the positive influence of the past events τ_i on the current value of the intensity process (φ is called the kernel).

Simple Hawkes process considered here

Hawkes (1971) proposes an exponential kernel $\varphi(t) = \sum_{j=1}^{P} \alpha_j e^{-\beta_j t} \mathbb{1}_{\mathbb{R}_+}(t)$, so that the intensity of the model becomes

$$\lambda(t) = \lambda_0(t) + \int_0^t \sum_{j=1}^{P} \alpha_j e^{-\beta_j (t-s)} dN_s = \lambda_0(t) + \sum_{\tau_i < t} \sum_{j=1}^{P} \alpha_j e^{-\beta_j (t-\tau_i)}.$$

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The simplest version with $P = 1$ and $\lambda_0(t)$ constant is defined as

$$\lambda(t) = \lambda_0 + \int_0^t \alpha e^{-\beta (t-s)} dN_s = \lambda_0 + \sum_{\tau_i < t} \alpha e^{-\beta (t-\tau_i)}.$$

Sample path of a 1D-Hawkes process

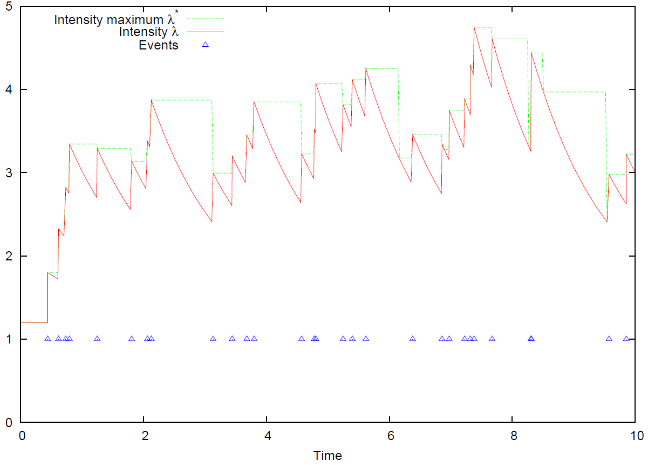

Figure: Simulation of a one-dimensional Hawkes process with parameters $P = 1$, $\lambda_0 = 1.2$, $\alpha = 0.6$, $\beta = 0.8$.

Stationarity : General case

Assuming stationarity gives $\mathbb{E}[\lambda(t)] = \mu$ constant. Thus,

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which gives

$$\mu = \frac{\lambda_0}{1 - \int_0^\infty \varphi(u)du}.$$

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Stationarity condition for a 1D-Hawkes process.

$$\sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} < 1.$$

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Average intensity of a stationary process : for the one-dimensional Hawkes process with $P = 1$, the unconditional expected value of the intensity process is given by

$$\mathbb{E}[\lambda(t)] = \frac{\lambda_0}{1 - \alpha/\beta}.$$

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Thinning procedure

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Basic thinning theorem. Consider a one-dimensional non-homogeneous Poisson process $(N_t^*)_{t\geq 0}$ with rate function $\lambda^*(t)$, so that the number of points $N_{T_0}^*$ in a fixed interval $(0, T_0]$ has a Poisson distribution with parameter $\mu_0^* = \int_0^{T_0} \lambda^*(s)ds$.

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Let $\tau^*_1, \ldots, \tau^*_{N^*_{T_0}}$ be the points of the process in the interval $(0, T_0]$. Suppose that for $0 \leq t \leq T_0$, $\lambda(t) \leq \lambda^*(t)$.

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For $i = 1, \dots, N^*(T_0)$, delete the points τ^*_i with probability $1 - \dfrac{\lambda(\tau^*_i)}{\lambda^*(\tau^*_i)}$.

Then the remaining points form a non-homogeneous Poisson process $(N_t)_{t\geq 0}$ with rate function $\lambda(t)$ in the interval $(0, T_0]$.

Simulation of a Hawkes process

Ogata (1981) proposes an algorithm for the simulation of Hawkes processes. Let us denote $\mathcal{U}_{[0,1]}$ the uniform distribution on the interval $[0, 1]$ and $[0, T]$ the time interval on which the process is to be simulated. We'll assume here that $P = 1$.

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   b. **New event** : Generate $U \sim \mathcal{U}_{[0,1]}$ and set $s \leftarrow s - \frac{1}{\lambda^*} \ln U$.
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   If $s \geq T$, **Then** go to the last step.
   c. **Rejection test** : Generate $D \sim \mathcal{U}_{[0,1]}$.
   If $D \leq \frac{\lambda(s)}{\lambda^*}$, **Then** $\tau_n \leftarrow s$ and go through the general routine again, **Else** update $\lambda^* \leftarrow \lambda(s)$ and try a new date at step b. of the general routine.

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4. **Output**: Retrieve the simulated process $(\tau_n)_n$ on $[0,T]$.

Example of simulations

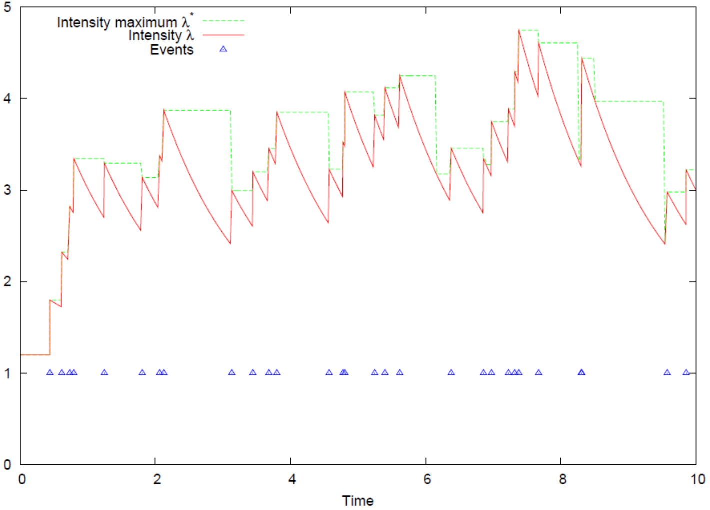

Figure: Simulation of a one-dimensional Hawkes process with parameters $P = 1$, $\lambda_0 = 1.2$, $\alpha_1 = 0.6$, $\beta_1 = 0.8$.

Testing the simulated process

For any consecutive events τ_{i-1} and τ_i

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&= \int_{\tau_{i-1}}^{\tau_i} \lambda_0(s)ds + \sum_{\tau_k\leq\tau_{i-1}} \frac{\alpha}{\beta}\left(e^{-\beta(\tau_{i-1}-\tau_k)} - e^{-\beta(\tau_i-\tau_k)}\right).
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This computation can be simplified with a recursive element. Let us denote

$$A_i = \sum_{\tau_k\leq\tau_i} e^{-\beta(\tau_i-\tau_k)}.$$

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Testing the simulated process: Time change property

Finally, the integrated density can be written for all $i \in \mathbb{N}^*$,

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where $(A_i)_{i \geq 0}$ is defined as above with $A_0 = 0$.

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The durations $\vartheta_i - \vartheta_{i-1} = \Lambda(\tau_{i-1}, \tau_i)$ are exponentially distributed.

Testing the simulated process

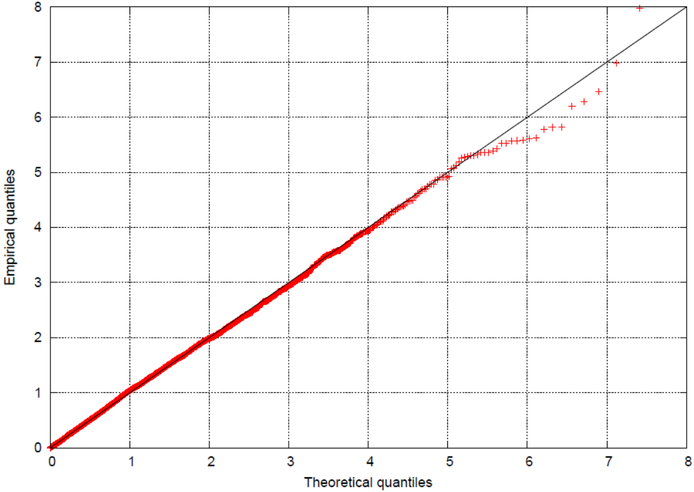

Figure: Quantile plot for one sample of simulated data of a one-dimensional Hawkes process with parameters $P = 1$, $\lambda_0 = 1.2$, $\alpha_1 = 0.6$, $\beta_1 = 0.8$, on an interval $[0, 10000]$.

Outline

Computation of the log-likelihood function

The log-likelihood of a simple point process N with intensity λ is written

$$\ln \mathcal{L}((N_t)_{t\in[0,T]}) = \int_0^T (1-\lambda(s))ds + \int_0^T \ln \lambda(s) dN_s,$$

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Direct computation of $\Lambda(0,\tau_n)$ yields to

$$\begin{aligned}\ln \mathcal{L}((\tau_i)_{i=1,\dots,n}) &= \tau_n - \int_0^{\tau_n} \lambda_0(s)ds - \sum_{i=1}^{n} \frac{\alpha}{\beta}\left(1 - e^{-\beta(\tau_n-\tau_i)}\right) \\ &\quad + \sum_{i=1}^{n} \ln\left(\lambda_0(\tau_i)+\alpha R_i\right).\end{aligned}$$

Properties of the maximum-likelihood estimator

Ogata (1978) shows that for a stationary one-dimensional Hawkes process with constant λ_0 and $P = 1$, the maximum-likelihood estimator $\hat{\theta}^T = (\hat{\lambda}_0, \hat{\alpha}_1, \hat{\beta}_1)$ is

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Numerical estimation of a simulated process

T	λ_0	α	β
100	1.210 (0.370)	0.588 (0.164)	0.833 (0.442)
1000	1.185 (0.133)	0.590 (0.044)	0.787 (0.068)
10000	1.204 (0.045)	0.602 (0.016)	0.804 (0.023)
100000	1.202 (0.014)	0.600 (0.004)	0.800 (0.007)
True values	1.200	0.600	0.800

Table: Maximum likelihood estimation of a one-dimensional Hawkes process on simulated data. Each estimation is the average result computed on 100 samples of length $[0, T]$. Standard deviations are given in parentheses. These results are obtained with a simple Nelder-Mead simplex algorithm.

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Multidimensionnal Hawkes process

Let $M \in \mathbb{N}^*$. Let $\{(\tau_i^m)_i\}_{m=1,\dots,M}$ be a M-dimensional point process. We will denote $N_t = (N_t^1, \dots, N_t^M)$ the associated counting process.

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Definition. A multidimensional Hawkes process is defined with intensities λ^m, $m = 1, \dots, M$ given by

$$\lambda^m(t) = \lambda_0^m(t) + \sum_{n=1}^{M} \int_0^t \sum_{j=1}^{P} \alpha_j^{mn} e^{-\beta_j^{mn}(t-s)} dN_s^n,$$

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i.e. in its simplest version with $P = 1$ and $\lambda_0^m(t)$ constant

$$\begin{aligned} \lambda^m(t) &= \lambda_0^m + \sum_{n=1}^{M} \int_0^t \alpha^{mn} e^{-\beta^{mn}(t-s)} dN_s^n \\ &= \lambda_0^m + \sum_{n=1}^{M} \sum_{\tau_i^n < t} \alpha^{mn} e^{-\beta^{mn}(t-\tau_i^n)}. \end{aligned}$$

Stationarity condition

Set $P = 1$ to simplify the notations. Using vectorial notation, we have

$$\lambda(t) = \lambda_0 + \int_0^t \Phi(t-s)dN_s, \quad \text{where} \quad \Phi(t) = \left(\alpha^{mn} e^{-\beta^{mn} t}\right)_{m,n=1,\dots,M}.$$

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We recall that the spectral radius of the matrix Φ is defined as

$$\rho(\Phi) = \max_{a \in \mathcal{S}(\Phi)} |a|, \quad \text{where } \mathcal{S}(\Phi) \text{ denotes the set of all eigenvalues of } \Phi.$$

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is thus the total intensity of the multivariate process and we set $I^0 = 0$. The algorithm is then rewritten as follows.

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1. **Initialization** : Set $i \leftarrow 1$, $i_1 \leftarrow 1, \ldots, i_M \leftarrow 1$ and $I^* \leftarrow I^M(0) = \sum_{i=1}^{M} \lambda_0^i(0)$.

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   a. **Update maximum intensity:** Set $I^* \leftarrow I^M(t_{i-1}) + \sum_{n=1}^{M} \sum_{j=1}^{P} \alpha_j^{nn_0}$.
   b. **New event:** Generate $U \sim \mathcal{U}_{[0,1]}$ and set $s \leftarrow s - \frac{1}{I^*} \ln U$. **If** $s > T$, **Then** go to the last step.

Simulation of a multivariate Hawkes process

1. **Initialization** : Set $i \leftarrow 1$, $i_1 \leftarrow 1, \ldots, i_M \leftarrow 1$ and $I^* \leftarrow I^M(0) = \sum_{i=1}^{M} \lambda_0^i(0)$.
2. **First event** : Generate $U \sim \mathcal{U}_{[0,1]}$ and set $s \leftarrow -\frac{1}{I^*} \ln U$.
   a. **If** $s > T$ **Then** go to last step.
   b. **Attribution Test** : Generate $D \sim \mathcal{U}_{[0,1]}$ and set $\tau_1^{n_0} \leftarrow s$ where n_0 is such that $\frac{I^{n_0-1}(0)}{I^*} < D \leq \frac{I^{n_0}(0)}{I^*}$.
   c. Set $\tau_1 \leftarrow \tau_1^{n_0}$.
3. **General routine** : Set $i^{n_0} \leftarrow i^{n_0} + 1$ and $i \leftarrow i + 1$.
   a. **Update maximum intensity:** Set $I^* \leftarrow I^M(t_{i-1}) + \sum_{n=1}^{M} \sum_{j=1}^{P} \alpha_j^{nn_0}$.
   b. **New event:** Generate $U \sim \mathcal{U}_{[0,1]}$ and set $s \leftarrow s - \frac{1}{I^*} \ln U$. **If** $s > T$, **Then** go to the last step.
   c. **Attribution-Rejection test:** Generate $D \sim \mathcal{U}_{[0,1]}$. **If** $D \leq \frac{I^M(s)}{I^*}$, **Then** set $\tau_{i_0^n}^{n_0} \leftarrow s$ where n_0 is such that $\frac{I^{n_0-1}(s)}{I^*} < D \leq \frac{I^{n_0}(s)}{I^*}$, and $\tau_i \leftarrow \tau_{i^{n_0}}^{n_0}$ and go through the general routine again, **Else** update $I^* \leftarrow I^M(s)$ and try a new date at step b. of the general routine.

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4. **Output**: Retrieve the simulated process $((\tau_i^n)_i)_{n=1,\ldots,M}$ on $[0, T]$.

Sample paths of a bivariate Hawkes process

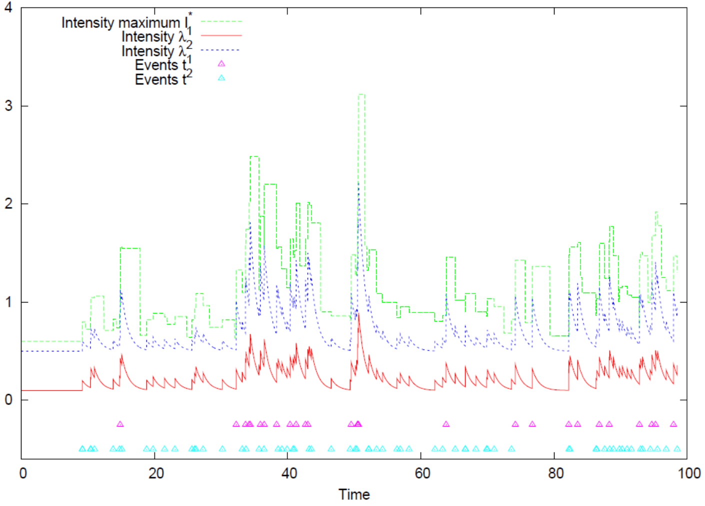

Figure: Simulation of a two-dimensional Hawkes process with P = 1 and $\lambda_0^1 = 0.1$, $\alpha_1^{11} = 0.2$, $\beta_1^{11} = 1.0$, $\alpha_1^{12} = 0.1$, $\beta_1^{12} = 1.0$, $\lambda_0^2 = 0.5$, $\alpha_1^{21} = 0.5$, $\beta_1^{21} = 1.0$, $\alpha_1^{22} = 0.1$, $\beta_1^{22} = 1.0$.

Testing the simulated data

The integrated intensity of the m-th coordinate of a multidimensional Hawkes process between two consecutive events t_{i-1}^m and t_i^m of type m is computed as

$$\Lambda^m(\tau_{i-1}^m, \tau_i^m) = \int_{\tau_{i-1}^m}^{\tau_i^m} \lambda^m(s)ds$$

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&\quad + \int_{\tau_{i-1}^m}^{\tau_i^m} \sum_{n=1}^{M} \sum_{\tau_{i-1}^m \leq \tau_k^n < s} \alpha^{mn} e^{-\beta^{nm}(s-\tau_k^n)} ds
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&+ \sum_{n=1}^{M} \sum_{\tau_{i-1}^m \leq \tau_k^n < \tau_i^m} \frac{\alpha^{mn}}{\beta^{mn}} \left[1 - e^{-\beta^{nm}(\tau_i^m - \tau_k^n)} \right].
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\end{aligned}
$$

This computation can be simplified with a recursive element. Let us denote

$$
A_{i-1}^{mn} = \sum_{\tau_k^n < \tau_{i-1}^m} e^{-\beta^{nm}(\tau_{i-1}^m - \tau_k^n)}.
$$

We observe that

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&\quad + \sum_{\tau_{i-2}^m \le \tau_k^n < \tau_{i-1}^m} e^{-\beta^{mn}(\tau_{i-1}^m - \tau_k^n)}
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&\quad + \sum_{\tau_{i-2}^m \le \tau_k^n < \tau_{i-1}^m} e^{-\beta^{mn}(\tau_{i-1}^m - \tau_k^n)} \\
&= e^{-\beta^{mn}(\tau_{i-1}^m - \tau_{i-2}^m)} A_{i-2}^{mn} + \sum_{\tau_{i-2}^m \le \tau_k^n < \tau_{i-1}^m} e^{-\beta^{mn}(\tau_{i-1}^m - \tau_k^n)}.
\end{aligned}
$$

The integrated density can thus be written $\forall i \in \mathbb{N}^*$,

$$\begin{aligned}
\Lambda^m(\tau_{i-1}^m, \tau_i^m) &= \int_{\tau_{i-1}^m}^{\tau_i^m} \lambda_0^m(s)ds \\
&\quad + \sum_{n=1}^{M} \frac{\alpha^{mn}}{\beta^{mn}} \left[\left(1 - e^{-\beta^{mn}(\tau_i^m - \tau_{i-1}^m)} A_{i-1}^{mn}\right) \right. \\
&\quad \left. + \sum_{\tau_{i-1}^m \leq \tau_k^n < \tau_i^m} \left(1 - e^{-\beta^{mn}(\tau_i^m - \tau_k^n)}\right) \right].
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where A is defined as previously with $A_0^{mn} = 0$.

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Time change property. As for the one-dimensional case, the durations $\vartheta_i^m - \vartheta_{i-1}^m = \Lambda^m(\tau_{i-1}^m, \tau_i^m)$ are exponentially distributed with parameter 1. See *e.g.* Bowsher (2007).

Testing the simulated data

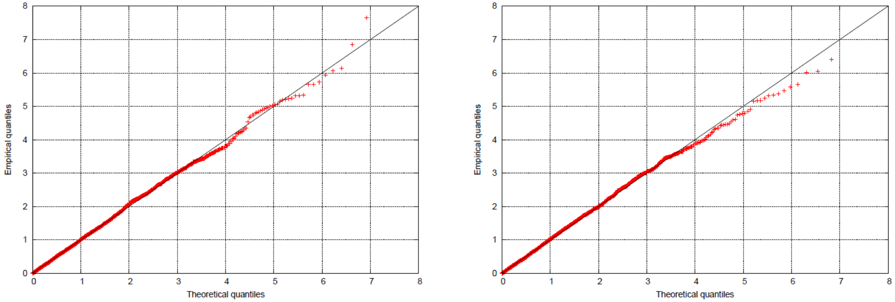

Figure: Quantile plots for one sample of simulated data of a two-dimensional Hawkes process with $P = 1$ and parameters given as in the simulation. *(Left)* $m = 1$. *(Right)* $m = 2$.

Outline

Computation of the log-likelihood function

The log-likelihood of a multidimensional Hawkes process can be computed as the sum of the likelihood of each coordinate, *i.e.* is written

$$\ln \mathcal{L}(\{\tau_i\}_{i=1,\ldots,N}) = \sum_{m=1}^{M} \ln \mathcal{L}^m(\{\tau_i\}),$$

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In the case of a multidimensional Hawkes process, denoting $\{\tau_i\}_{i=1,\dots,N}$ the ordered pool of all events $\{\{\tau_i^m\}_{m=1,\dots,M}\}$, this log-likelihood can be computed as

$$\begin{aligned} \ln \mathcal{L}^m(\{\tau_i\}) &= T - \Lambda^m(0,T) \\ &\quad + \sum_{i=1}^{N} z_i^m \ln \left(\lambda_0^m(\tau_i) + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{\tau_k^n < \tau_i} \alpha_j^{mn} e^{-\beta_j^{mn}(\tau_i - \tau_k^n)} \right), \end{aligned}$$

where z_i^m is equal to 1 if the event τ_i is of type m, 0 otherwise.

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$$R_l^{mn} = \sum_{\tau_k^n < \tau_l^m} e^{-\beta^{mn}(\tau_l^m - \tau_k^n)}$$

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$$= \begin{cases} e^{-\beta^{mn}(\tau_l^m - \tau_l^m - 1)} R_{l-1}^{mn} + \displaystyle\sum_{\tau_{l-1}^m \leq \tau_k^n < \tau_l^m} e^{-\beta^{mn}(\tau_l^m - \tau_k^n)} & \text{if} \quad m \neq n, \\ e^{-\beta^{mn}(\tau_l^m - \tau_l^m - 1)}(1 + R_{l-1}^{mn}) & \text{if} \quad m = n. \end{cases}$$

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The final expression of the log-likelihood may be written

$$\begin{aligned} \ln \mathcal{L}^m(\{\tau_i\}) &= T - \sum_{i=1}^{N}\sum_{n=1}^{M} \frac{\alpha^{mn}}{\beta^{mn}}\left(1-e^{-\beta^{mn}(T-\tau_i)}\right) \\ &\quad + \sum_{\tau_l^m} \ln\left(\lambda_0^m(\tau_l^m) + \sum_{n=1}^{M} \alpha^{mn} R_l^{mn}\right), \end{aligned}$$

where R_l^{mn} is defined above and $R_0^{mn}=0$.

Numerical estimation of a simulated process

T	λ_0^1	α^{11}	$\beta 11$	α^{12}	β^{12}	λ_0^2	α^{21}	β^{21}	α^{22}	β^{22}
100	0.614	0.510	0.369	1.482	1.710	0.518	0.337	0.600	1.605	2.595
	(0.372)	(0.268)	(0.269)	(1.216)	(3.172)	(0.272)	(0.206)	(0.365)	(2.051)	(6.586)
500	0.516	0.505	0.268	1.043	0.865	0.518	0.264	0.507	0.814	1.048
	(0.112)	(0.085)	(0.080)	(0.214)	(0.479)	(0.120)	(0.080)	(0.084)	(0.278)	(0.221)
1000	0.507	0.492	0.254	1.018	0.761	0.513	0.255	0.488	0.794	1.003
	(0.079)	(0.054)	(0.052)	(0.122)	(0.203)	(0.092)	(0.052)	(0.061)	(0.387)	(0.152)
True	0.500	0.500	0.250	1.000	0.750	0.500	0.250	0.500	0.750	1.000

Table: Maximum likelihood estimation of a two-dimensional Hawkes process on simulated data. Each estimation is the average result computed on 100 samples of length $[0, T]$. Standard deviations are given in parentheses.

Outline

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A one-dimensional price model

Bacry, Delattre, Hoffmann & Muzy (2013) propose a one-dimensional case where the price p is written

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$$\lambda_1(t) = \lambda_0 + \int_{-\infty}^{t} \alpha e^{-\beta(t-s)} dN_s^2,$$

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- No self-excitation of upward (resp. downward) jumps on following upward (resp. downward) jumps.
- Only cross-excitation terms are kept, enforcing the mean-reversion empirically observed on the price p.
- Cross-excitation is set to be symmetric.

An analytical expression for the variance of the price

A volatility signature plot plots the realized variance as a function of the sampling period

$$RV(\tau) = \frac{1}{\tau} \sum_{i=1}^{I} (\hat{p}(i\tau) - \hat{p}((i-1)\tau))^2,$$

where I is the number of observations $\hat{p}$.

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where

$$\Lambda = \frac{2\lambda_0}{1-\alpha/\beta}, \quad \kappa = \frac{1}{1+\alpha/\beta} \quad \text{and} \quad \gamma = \alpha + \beta.$$

See Bacry et al. (2011), Appendix 1.

Likehood functions

Consequently the likelihood functions $\theta \to L_1(\theta)$ and $\theta \to L_2(\theta)$ generated by the observation of $N^1(t)$ and $N^2(t)$ over $[0, T]$ are given by

$$\begin{aligned} L_1(\theta) &= \sum_{0 \le \tau_i^{(1)} < N^1(T)} \ln\left(\lambda_0 + \sum_{0 \le \tau_j^{(2)} < N^2(T)} \alpha e^{-\beta(\tau_i^{(1)} - \tau_j^{(2)})}\right) \\ &\quad -(\lambda_0 - 1)T - \sum_{0 \le \tau_j^{(2)} < N^2(T)} \frac{\alpha}{\beta}\left(1 - e^{-\beta(T - \tau_j^{(2)})}\right), \end{aligned}$$

Likehood functions

$$L_2(\theta) = \sum_{0\le\tau_i^{(2)}<N^2(T)} \ln\left(\lambda_0 + \sum_{0\le\tau_j^{(1)}<N^2(T)} \alpha e^{-\beta(\tau_i^{(2)}-\tau_j^{(1)})}\right)$$
$$-(\lambda_0-1)T - \sum_{0\le\tau_j^{(1)}<N^1(T)} \frac{\alpha}{\beta}\left(1-e^{-\beta(T-\tau_j^{(1)})}\right).$$

and the MLE of θ based on the observation of $p(t)$ over $[0, T]$ is thus given by

$$\hat{\theta}_{MLE} = \mathrm{argmin}_\theta\left(L_1(\theta)+L_2(\theta)\right).$$

Of course, the minimization must be performed under the constraints

$$\lambda_0 > 0, \quad \alpha > 0, \quad \beta > 0, \quad \text{and the stability condition } \frac{\alpha}{\beta} < 1.$$

Fitting the volatility signature plot : Simulated data

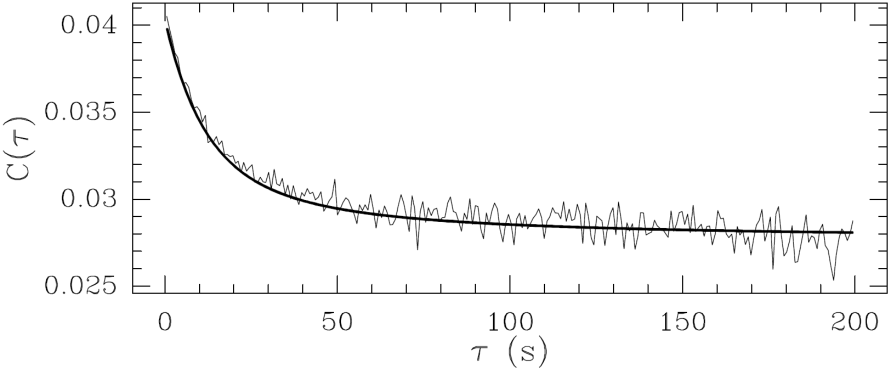

Figure: Numerical simulation of a 1D price model using mutually exciting Hawkes processes with parameters $\lambda_0 = 0.16$, $\alpha = 0.024$, $\beta = 0.11$. Estimated signature plot $C(\tau)$ with the corresponding expected analytical shape. τ is expressed in seconds. Reproduced from Bacry et al. (2011).

Fitting the volatility signature plot : Market data

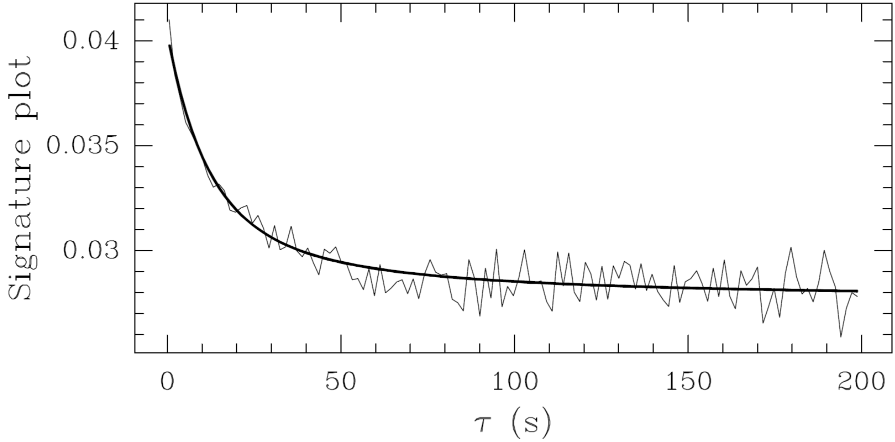

Figure: Euro-Bund contract on the 3rd of November 2009 (contract maturity 12/2009) from 8am to 10pm. Associated mean daily signature plot. The solid line represents the fit using the regression estimator for the 1D Hawkes model. Reproduced from Bacry et al. (2011).

Outline

A two-dimensional model

Bacry et al. (2013) also propose a bivariate version of the model

$$p^1(t) = N^1(t) - N^2(t),$$
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$$p^2(t) = N^3(t) - N^4(t),$$

in which $\mathbf{N} = (N^i)_{i=1,\dots,4}$ is a Hawkes process with intensity

$$\lambda(t) = \lambda_0 + \int_0^t \begin{pmatrix} 0 & \varphi^{12} & \varphi^{13} & 0 \\ \varphi^{12} & 0 & 0 & \varphi^{13} \\ \varphi^{31} & 0 & 0 & \varphi^{34} \\ 0 & \varphi^{31} & \varphi^{34} & 0 \end{pmatrix} (t-s) d\mathbf{N}_s,$$

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- prices p^1 and p^2 influence each other in a positive way, not a negative one ($\varphi^{14} = \varphi^{41} = \varphi^{23} = \varphi^{32} = 0$).

An explicit expression for the covariance matrix

For this model, Bacry et al. (2013) show that an explicit form of the correlation coefficient

$$\rho(\tau) = \mathrm{Corr}(p^1(t+\tau) - p^1(t), p^2(t+\tau) - p^2(t)) = \frac{C_{12}(\tau)}{C_{11}(\tau)},$$

where $C_{ij}(\tau) = \mathrm{Cov}(p^i(t+\tau) - p^i(t), p^j(t+\tau) - p^j(t))$, can be explicitly computed, although the expected result is quite cumbersome (see Bacry et al. (2011), Proposition 3.1).

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Moreover, they get

$$\rho(\tau) = \frac{R(Q_2 - Q_1)}{4\Lambda}\tau + O(\tau^2) \quad \text{as } \tau \to 0,$$

$$\rho(\tau) = \frac{2\Gamma_{13}(1+\Gamma_{12})}{1+\Gamma_{13}^2 + 2\Gamma_{12} + \Gamma_{12}^2} \quad \text{as } \tau \to \infty,$$

where $\Gamma_{ij} = \alpha^{ij}/\beta^{ij}$, $\Lambda = \lambda_0/(1-\Gamma_{12}-\Gamma_{13})$, $R = \beta\lambda_0//(\Gamma_{12}+\Gamma_{13}-1)$, $Q_1 = -\lambda_0(\Gamma_{12}^2 + \Gamma_{12} - \Gamma_{13}^2)/((((\Gamma_{12}+1)^2 - \Gamma_{13}^2)(1-\Gamma_{12}-\Gamma_{13}))$ and $Q_1 = -\lambda_0\Gamma_{13}/((((\Gamma_{12}+1)^2 - \Gamma_{13}^2)(1-\Gamma_{12}-\Gamma_{13}))$.

Addressing the Epps effect : Simulated data

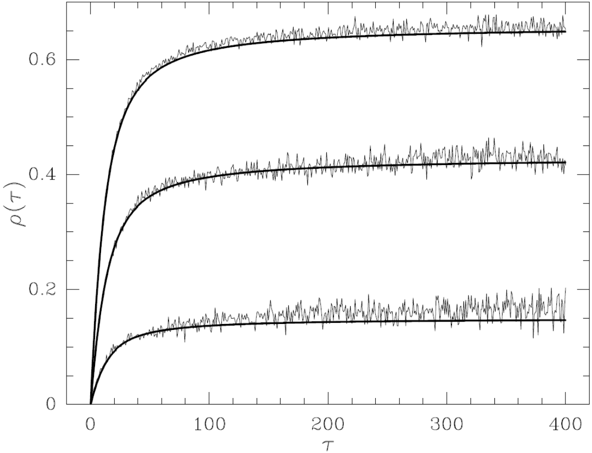

Figure: Epps effect for simulation samples. The estimated correlation coefficients are compared to the expected analytical curves (solid lines). From top top bottom the cross correlation between the two asset returns increases from 0.15 to 0.65. Reproduced from Bacry et al. (2011).

Addressing the Epps effect : Market data

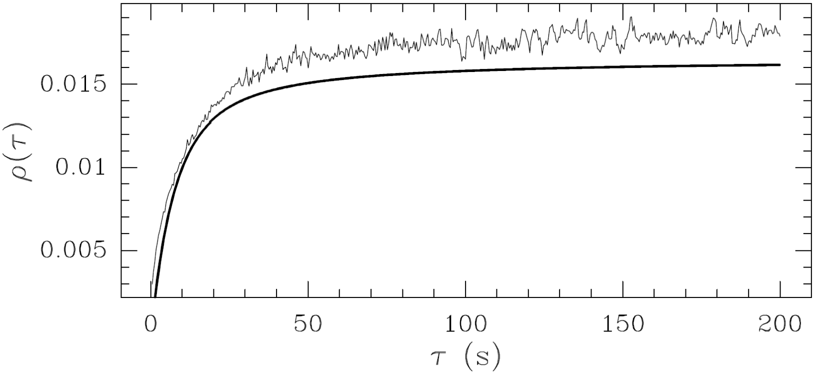

Figure: Epps effect between Euro-Bund et Euro-Bobl returns as measured by the covariance $C_{12}(\tau)$ between the two asset returns at different scales τ. The heavy lines represent a fit of the empirical curves according to the 2D Hawkes model. Reproduced from Bacry et al. (2011).

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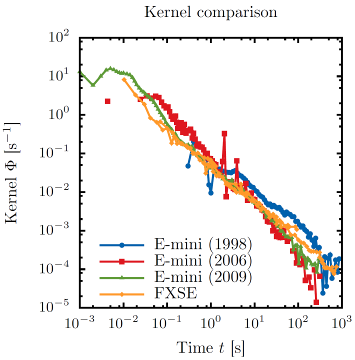

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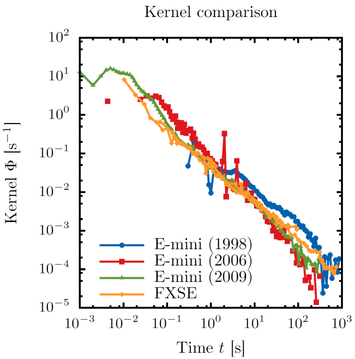

In [3], they use non-parametric estimation for the kernels to confirm this power-law behavior and the existence of non-negligible diagonal and anti-diagonal terms.

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Let $\varphi = \varphi_1 - \varphi_2$. Under some assumptions, the sequence of Hawkes-based price models $P_t^T = \frac{1}{T}\left(N^1(Tt) - N^2(Tt)\right)$, converges in law, for the Skorohod topology, toward a Heston-type process P on $[0,1]$ defined by

$$\begin{cases} dC_t = \left(\frac{2\mu}{\lambda} - C_t\right)\frac{\lambda}{m}dt + \frac{1}{m}\sqrt{C_t}dB_t^1, & C_0 = 0, \\ dP_t = \frac{1}{1 - ||\varphi||_1}\sqrt{C_t}dB_t^2, & P_0 = 0, \end{cases}$$

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This work can be seen as the very first step towards an "across scales" unified model that would fit both microstructure stylized facts of price (*i.e.*, point process with strong mean reversion) and "diffusive" stylized facts (volatility clustering and multifractality).

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They have shown that their model is consistent with empirical data by reproducing the signature plots of the considered assets and the behavior of the high-frequency pair correlation function.

Outline

Leverage effect

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where $\mathrm{Cov}(dP_{t-\tau}, \sigma_t^2)$ is the covariance between past returns and future volatility (lagged by τ) and, conversely, $\mathrm{Cov}(dP_t, \sigma_{t-\tau}^2)$ is the covariance between past volatility and future returns.

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can be understood by recognizing that large negative returns induce market panic, thereby increasing future volatility, while, high volatility does not necessarily lead to large future returns.

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Intuitively, the Zumbach effect captures the notion that a trending market (characterized by an increase of R^2_t over several days) heightens market participants' uncertainty, leading to increased volatility, while increased volatility does not necessarily result in a trending market.

Definition of QHawkes

A quadratic Hawkes process $(N_t)_{t\geq 0}$ is an inhomogenous Poisson process whose intensity is dependent on both past activity (dN_t) and past price returns defined as $dP_t := \varepsilon_t \psi dN_t$, where $\varepsilon_t = \pm 1$ is an unbiased random sign, independently chosen at each price change, and ψ is the tick size (size of elementary price changes).

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$$\lambda_t = \lambda_0 + \frac{1}{\psi}\int_{-\infty}^{t} L(t-s)dP_s + \frac{1}{\psi^2}\int_{-\infty}^{t}\int_{-\infty}^{t} Q(t-s,t-u)dP_s dP_u,$$

where L is called the leverage kernel and Q the quadratic feedback kernel. Note that L and Q must be such that the quadratic form in dP_s is positive definite.

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Since $dP_t^2 = \psi^2 dN_t$, a purely diagonal Q recovers the standard Hawkes kernel. New effects arise when considering non-diagonal contributions to Q.

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In this case, the intensity of the QHawkes becomes

$$\lambda_t = \lambda_0 + H_t + Z_t^2,$$

where $H_t = \int_{-\infty}^t \phi(t-s)dN_s$ represents the Hawkes component of the intensity, $N_t - N_s := \frac{1}{\psi^2}(P_t - P_{t-})^2$, whereas $Z_t = \frac{1}{\psi}\int_{-\infty}^t z(t-s)dP_s$ represents the trend-induced (Zumbach) component.

Multivariate QHawkes processes

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- Multivariate QHawkes processes seem to consistently extend the relevant features of the univariate QHawkes: fat tails of the returns distribution, volatility clustering, cross-time-reversal asymmetry.

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First Level-I model with Hawkes process

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Large (2007) formalized the ability of LOB to replenish after being depleted by a large trade (book resiliency) by quantifying it by the way large trades alter future intensities of order occurrences. To this end, he introduced the response kernel

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With data from LSE, he modelled as a $10D$-Hawkes processes:

- MOs and LOs that move the mid-prices (4 components with bid/ask),
- MOs and LOs that leave the book unchanged (4 components),
- cancel orders (2 components).

Large used MLE within the class of exponential kernels in order to estimate Ψ defined by the causal solution of $\Phi(t) + \Psi(t) * \Phi(t) = \Psi(t)$.

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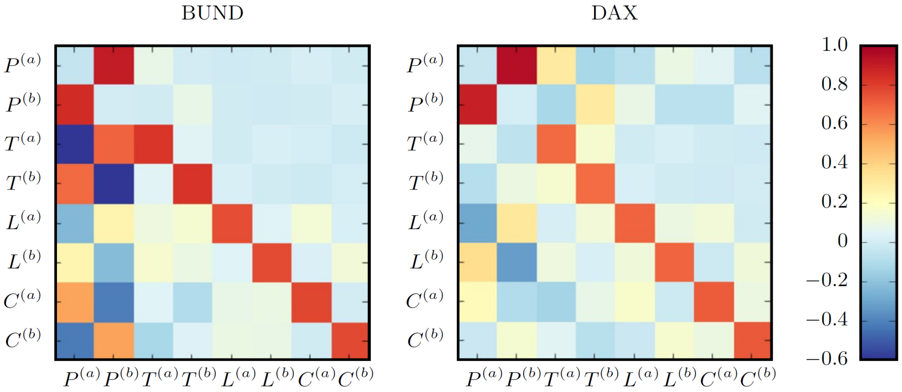
- $C^{(a)}$ (resp. $C^{(b)}$) counts the number of cancel orders at the ask (resp. bid) that do not move the price.

Results of this second model

- The book event dynamics is mainly self-exciting, except for the mid-price changes for which cross-excitation effects are strongly dominating (see the following figure): one can see that the resulting matrix is mainly diagonal except in the price sub-block, that is anti-diagonal.
- Price changes events are mainly triggered by price change events, but this effect is mostly anti-diagonal.
- The market is highly endogenous and all the dominating kernels are slowly decreasing, well described by a power-law behavior.
- They also provided evidence of some inhibitory effects which result from negative values of some Hawkes kernels. For example it was observed that, for a large tick asset, an upward price move not only triggers forthcoming trades at bid but also inhibits trades on the ask side.

Color map of the matrix of Hawkes kernel norms



Empirical determination of the matrix of Hawkes kernel norms in the 8D-model of level-I book events. This figure is reproduced from Bacry and *al.* (2016).

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All the volumes of both limit and market orders are i.i.d. and exponentially distributed variables. Hawkes processes use exponential kernels which are estimated parametrically using MLE.

Results of this model

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- Inter-time between a given MO and the first arrival time of a LO after that MO is much better reproduced by a $2D$-Hawkes (MM+LL+LM) model with self-exciting kernels for both market (MM) and limit orders (LL) and a single cross-exciting kernel which corresponds to the influence of past MOs on future LOs (LM).

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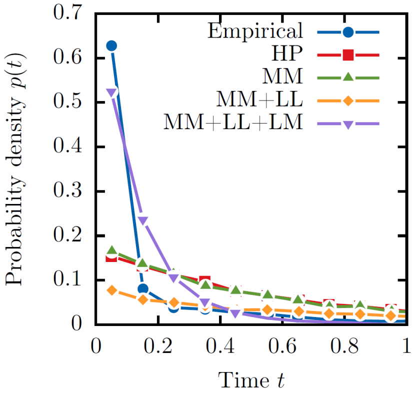
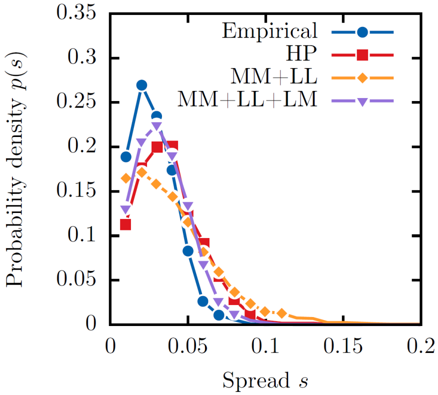
- Finally the right panel of next figure shows the so-obtained distribution of the bid-ask spread for empirical data, the pure Homogenous Poisson (HP) model, the MM+LL Hawkes model and the MM+LL+LM model.

Empirical density function





Empirical density function of the distribution of the duration between a given market order and the first arrival of a limit order after that market order. Reproduced from Muni Toke (2010).

Full order book model II

Abergel and Lu (2018) introduced a point-process-based order book model, building on the 12 event types

$$E = \{L^0_{buy}, L^0_{sell}, C^0_{buy}, C^0_{sell}, M^0_{buy}, M^0_{sell}, \\ L^1_{buy}, L^1_{sell}, C^1_{buy}, C^1_{sell}, M^1_{buy}, M^1_{sell}\},$$

where L stands for LO, M for MO, C for cancellation, 0 for orders that does not change the mid-price and 1 for orders that impact the mid-price.

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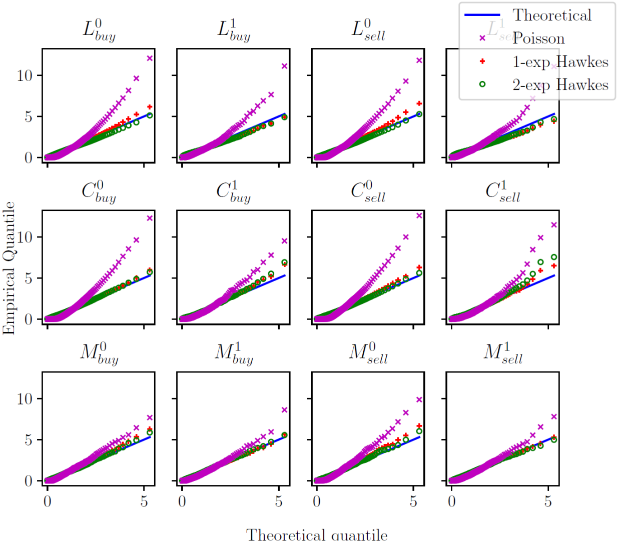
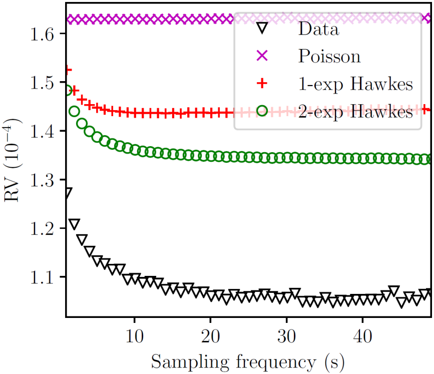
- They consider both linear and non-linear Hawkes processes with exponential kernels.
- They consider two criteria to evaluate the performances of their model: a goodness-of-fit criterion for the distribution of forward recurrence times, and a criterion based on the signature plot generated by the model.

Q-Q plot goodness of fit tests of linear order book models.



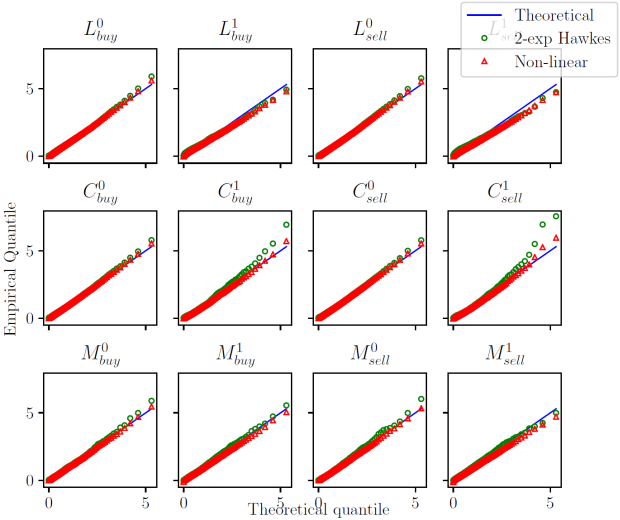
Reproduced from Abergel and Lu (2018).

Signature plots of linear order book models.



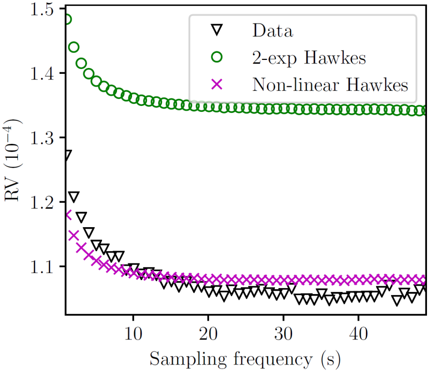
Mean signature plots of simulated price compared with real data for ALLIANZ SE. Reproduced from Abergel and Lu (2018).

Q-Q plot goodness of fit tests of non-linear order book models.



Reproduced from Abergel and Lu (2018).

Signature plots of non-linear order book models.



Mean signature plots of simulated price compared with real data for ALLIANZ SE. Reproduced from Abergel and Lu (2018).

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