Binomial Trees in Option Pricing—History, Practical Applications and Recent Developments

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Abstract We survey the history and application of binomial tree methods in option pricing. Further, we highlight some recent developments and point out problems for future research.

1 Introduction

In many disciplines, there is the classical question on which came first, egg or hen; but not so in the history of binomial option valuation. There is no denying the fact that the diffusion model underlying the famous Black-Scholes formula (see Black and Scholes 1973) triggered the development of the binomial approach to option pricing. At first sight it seems surprising that the binomial approach originates from the Black-Scholes model although the mathematics behind diffusion models are clearly much more involved than that behind the discrete-time and finite state space binomial models (the reason why we interpret the Black-Scholes model as the *hen* and the simpler binomial model as the *egg*). To understand why the hen came first, we need to recognize option valuation as a discipline that brings together mathematical modeling skills and economic interpretations of real-world markets.

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The Black-Scholes formula caused a shock amongst the economists at the time of its introduction. The economic ideas underlying the Black-Scholes approach, such as the principles of risk-neutrality and riskless portfolios, shook the theory of option pricing to its core. However, its involved mathematical background based on diffusion models might have appeared too academic or even awkward. This motivated various economists to search for a simpler modeling framework that preserves the economically relevant properties of the Black-Scholes framework but that is at the same time more easily accessible. The binomial approach to option pricing grew out of a discussion between M. Rubinstein and W.F. Sharpe at a conference in Ein Borek, Israel (see Rubinstein 1992 for the historical background). They realized that the economic idea behind the Black-Scholes model can be reduced to the following principle: If an economy incorporating three securities can only attain two future states, one such security will be redundant; i.e. each single security can be replicated by the other two, a fact later referred to as market completeness. With this insight at hand, it was obvious that one should introduce such a two-state model and verify that the economic properties of the Black-Scholes diffusion approach are preserved. This was the birth of binomial option pricing.

In this survey, we will first explain how to use binomial trees for option pricing in the corresponding discrete-time financial market. However, in practical applications, binomial trees are preferably used as numerical approximation tools for pricing options in more complex, continuous-time stock market models. We will present early approaches to binomial trees: the models suggested by Cox et al. (1979) and by Rendleman and Bartter (1979). In Black-Scholes settings, the application of the binomial approach to numerical option pricing can be justified by Donsker's Theorem on random walk approximations to a Brownian motion. Donsker's Theorem implies that as the period length tends to zero, the sequence of corresponding binomial models (appropriately scaled in time) converges weakly to a geometric Brownian motion, which underlies the Black-Scholes stock price model (provided that the first two moments of the one-period log-returns are matched). The application of binomial models as numerical pricing tools will be explained in detail. We will focus on aspects of practical relevance such as the convergence behavior of binomial estimates to Black-Scholes option prices, the speed of convergence and the algorithmic implementation. In particular, we discuss how to generalize the approximation by binomial models to option pricing in the multi-asset Black-Scholes setting.

2 Option Pricing and Binomial Tree Models: the Single Asset Case

An n-period binomial tree is a simple stochastic model for the dynamics of a stock price evolving over time. More precisely, it is a discrete-time stochastic process $\{S^{(n)}(i), i \in \{0, 1, \ldots, n\}\}$ such that

$$S^{(n)}(i+1) = \begin{cases} uS^{(n)}(i), & \text{if the price increases from period } i \text{ to } i+1, \\ dS^{(n)}(i), & \text{if the price decreases from period } i \text{ to } i+1, \end{cases}$$

where we require $u > d$ and $S^{(n)}(0) = s$. By our convention $u > d$, u is the favorable one-period return. The time spacing is assumed to be equidistant, so that each period has length $\Delta t = T/n$, where T is the time horizon. We also assume that at each state ("node") of the tree, we have the same probability $p \in (0, 1)$ to achieve the favorable one-period return u.

If we assume that in addition to trading this stock, the investor can also invest in a bank account with a continuously compounded interest rate r (i.e. investment grows by the factor $e^{r\Delta t}$ per period), we will require the no-arbitrage relation

$$u > e^{r\Delta t} > d. \tag{1}$$

If the above relation is violated, one can generate money without investing own funds by either selling the stock short (in the case $u \leq e^{r\Delta t}$) or financing a stock purchase by a credit (in the case of $d \geq e^{r\Delta t}$). In the following, we assume that the market is arbitrage-free (i.e. that (1) holds).

In the highly simplified financial market introduced above, an option is a functional $B = f(S^{(n)}(i), i = 1, \ldots, n)$ of the path of the stock price process. The owner of the option receives the payment B at the time horizon T (the maturity). As the final payment is a function of the stock price process, it is not known at the purchasing date. Consequently, trading the option can be identified with a bet on the evolution of the stock price $S^{(n)}$. Analogous to the Black-Scholes setting, the "fair" option price in the binomial model can be obtained via *the principle of replication*, i.e. one determines the costs required to set up a trading strategy in the stock and the riskless investment opportunity that will realize the same final payment B as received by holding the option (independently of the realized stock price movements!). We have the following basic result (see Bjoerk 2004):

Theorem 1 (Risk-Neutral Valuation and Replication). *Each option B in an n-period binomial model can be replicated by an investment strategy in the stock and the bond. The initial costs of this strategy determine the option price and are given by*

$$c_0 = E_{Q^{(n)}}\big(e^{-rT}B\big),$$

where the measure $Q^{(n)}$ is the product measure of the one-period transition measures $Q_i^{(n)}$ which are determined by

$$Q_i^{(n)}\left(\frac{S^{(n)}(i+1)}{S^{(n)}(i)} = u\right) = q = \frac{\exp(r\Delta t) - d}{u-d},$$

and for which we have

$$S^{(n)}(0) = E_{Q^{(n)}}\left(e^{-rT}S^{(n)}(n)\right). \tag{2}$$

Equation (2) shows that under $Q^{(n)}$ the expected relative return of the stock and the bond coincide. This motivates calling $Q^{(n)}$ the *risk-neutral measure* (note that it can easily be verified that $Q^{(n)}$ is the unique equivalent probability measure with this property). The risk-neutral probability q gives us the market view on the likelihood that the favorable one-period return u is attained. It can be different from the physical probability p. Then, if $E^{(n)}(e^{-rT}B)$ is computed with respect to p, we have $E^{(n)}(e^{-rT}B) \neq E_{Q^{(n)}}(e^{-rT}B)$, where $E_{Q^{(n)}}(e^{-rT}B)$ is the option price.

Note. The above result is identical to the result in the Black-Scholes setting: The underlying market is complete (i.e. every (suitably integrable) final payment can be replicated by appropriate trading in the underlying and the riskless investment) and the resulting option price is obtained as the net present value of the option payment under the risk-neutral measure. The risk-neutral measure is equivalent to the physical measure for the stock price evolution. As under the risk-neutral measure the corresponding discounted price processes of both assets are martingales, it is also called the equivalent martingale measure.

As seen above, the binomial approach leads to a modeling framework for option pricing that is technically easy and contains economically meaningful insights. However, the question remains whether the binomial model is in any reasonable way related to the Black-Scholes stock price model for which the stock price $\{S(t), t \in [0, T]\}$ is assumed to follow a geometric Brownian motion; that is

$$S(t) = s \cdot \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)$$

with $W(t)$ a one-dimensional Brownian motion (under the risk-neutral measure Q associated with the continuous-time financial market) and $\sigma > 0$ a given constant describing the volatility of the stock price movements. The above question can be made more precise in two different ways: As the period length tends to zero,

- do we have (weak) convergence of the stock price paths in the sequence of increasing binomial models to the given geometric Brownian motion?
- does the sequence of binomial option prices $(E_{Q^{(n)}}(e^{-rT}B))_n$ converge to the corresponding option price in the Black-Scholes model?

The answers to these questions are intimately related to the concept of weak convergence of the corresponding stochastic processes. In particular, if we are only interested in the terminal value of the stock $S(T)$, the questions are answered by the classical Central Limit Theorem: Let X_n denote the (random) number of up-movements of the stock price in an n-period binomial model. We obviously have

$$X_n \sim B(n, p),$$

where $B(n, p)$ denotes the Binomial distribution with n trials and success probability p. Rewriting the stock price in the n-period binomial model as

$$S^{(n)}(n) = s \cdot u^{X_n} \cdot d^{n-X_n} = s \cdot e^{X_n \cdot \ln\left(\frac{u}{d}\right) + n \cdot \ln(d)},$$

using the choice $p = 1/2$, $b = r - \frac{1}{2}\sigma^2$ and

$$u = e^{b\Delta t + \sigma\sqrt{\Delta t}}, \qquad d = e^{b\Delta t - \sigma\sqrt{\Delta t}},$$

the Central Limit Theorem implies that

$$\begin{aligned} S^{(n)}(n) &= s \cdot \exp\left(bT + \sigma\sqrt{T}\left(\frac{2X_n - n}{\sqrt{n}}\right)\right) \\ &\xrightarrow{D} s \cdot \exp\left(bT + \sigma W(T)\right) = S(T). \end{aligned}$$

Hence, for the above parameter specifications, the terminal stock price in the binomial model $S^{(n)}(n)$ converges in distribution to the terminal value in the Black-Scholes model $S(T)$. Furthermore, provided the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies suitable regularity conditions, the sequence $(E^{(n)}(e^{-rT}g(S^{(n)}(n))))_n$ obtained along the increasing binomial models converges to the quantity $E_Q(e^{-rT}g(S(T)))$ obtained in the Black-Scholes model; for instance, it clearly suffices that g is bounded and continuous. Yet $E_Q(e^{-rT}g(S(T)))$ is the Black-Scholes price for an option with payment $B = g(S(T))$. Consequently, for path-independent options (i.e. options that depend only on the terminal stock price), the above argument allows us to apply the binomial model specified above as a numerical valuation tool to approximate the option price in the Black-Scholes model. Of course, this is useful in practical applications if an explicit pricing formula is not known in the Black-Scholes setting. However, for the given parameter specifications, the sequence of binomial option prices $(E_{Q^{(n)}}(e^{-rT}g(S^{(n)}(n))))_n$ does in general not converge to the Black-Scholes option price (!). Hence, we observe that for numerical option pricing, it is only relevant whether the terminal distribution of the binomial model approximates the lognormal distribution specifying the terminal stock price $S(T)$. By contrast, it is irrelevant whether the corresponding probability p is determined according to the risk-neutral measure.

To introduce a general approximation technique that also works for option payments depending on the entire path of the stock price process, we have to invoke the concept of weak convergence of stochastic processes: Assume that the binomial model is such that the first two moments of the one-period log-returns of the stock price process S are matched. Then it follows from Donsker's Theorem (see e.g. Billingsley 1968) that (after linear interpolation) the binomial stock price process converges weakly to the geometric Brownian motion underlying the Black-Scholes stock price model. Hence, we have an affirmative answer to our first question whether the two stock price models can be related to one another. Furthermore, it follows from the definition of weak convergence that if the payoff function is bounded and continuous, the corresponding sequence $(E^{(n)}(e^{-rT}f(S^{(n)}(i), i = 1, \ldots, n)))_n$ converges to the option price in the Black-Scholes model. In fact, binomial option valuation can be justified for most common types of traded options; but this issue will not be addressed in this survey.

Note. According to the above arguments, the binomial method can be applied to numerical valuation of options in the Black-Scholes model. In this context, it is irrelevant whether the probability p coincides with the risk-neutral probability q (compare the above example where $p \neq q$). We only require that p be chosen such that the first two moments of the one-period log-returns are matched, so that the binomial stock price model approximates the Black-Scholes stock price model.

Of course, there are many possibilities to satisfy the moment matching conditions. The first suggestions were made by Cox, Ross and Rubinstein (CRR tree) and by Rendleman and Bartter (RB tree). The CRR tree is determined by the parameter specifications

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = 1/u, \qquad p = \frac{1}{2}\left(1 + \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{\sigma}\sqrt{\Delta t}\right).$$

Note that the probability p of an up-movement is only well-defined provided the grid size is sufficiently small; to be precise, we need that

$$n > \frac{(r - \frac{1}{2}\sigma^2)^2}{\sigma^2} T.$$

Note further that under the above specification of parameters, the second moment of the log-returns in the Black-Scholes model is only matched asymptotically; i.e. if grid size tends to zero. However, due to Slutsky's Theorem, it is clear that weak convergence is preserved. The CRR model is such that the log-tree (i.e. the tree containing the log-prices in the binomial model) is symmetric around the initial price $S^{(n)}(0)$. Upward or downward tendencies in the log-prices of the Black-Scholes model are incorporated into the binomial model via the above choice of the probability p.

The RB tree is given by the example considered above; that is,

$$p = \frac{1}{2}, \qquad u = e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}, \qquad d = e^{(r-\frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}.$$

For this specification of parameters, the probabilities are automatically well-defined and symmetric. Upward or downward tendencies in the log-prices of the Black-Scholes model are incorporated into the discrete model via an appropriate form of the one-period returns u and d.

Both models ensure weak convergence to the Black-Scholes stock price model. Consequently, provided that the grid size is sufficiently small, the discounted expected value of the option payoff in the binomial models $E^{(n)}(e^{-rT} f(S^{(n)}(i); i = 1, \dots, n))$ approximates the corresponding Black-Scholes option price. However, for both methods, the computed discounted expected option payoff does in general not coincide with the discrete-time option price because $p \neq q$. As a consequence, these models do not admit a simple economic interpretation. However, if the real-world market is modeled according to Black-Scholes, they can be applied to numerical option valuation.

3 Binomial Trees in Action—Implementation, Problems and Modifications

The binomial approach offers an attractive numerical pricing method because it can be implemented in form of an efficient backward algorithm. More precisely, for path-independent options with payment $B = f(S^{(n)}(i), i = 1, \dots, n) = f(S^{(n)}(n))$, we have the following backward recursion:

Algorithm. Backward induction in the CRR tree

1. Set $V^{(n)}(T, S^{(n)}(n)) = f(S^{(n)}(n))$.
2. For $i = n-1, \dots, 0$ do

$$\begin{aligned} V^{(n)} &\left(i \cdot \Delta t, S^{(n)}(i)\right) \\ &= \left[pV^{(n)}\left((i+1) \cdot \Delta t, uS^{(n)}(i)\right) \right. \\ &\quad \left. + (1-p)V^{(n)}\left((i+1) \cdot \Delta t, \frac{1}{u}S^{(n)}(i)\right)\right] \cdot e^{-r\Delta t}. \end{aligned}$$

3. Set $E^{(n)}(e^{-rT}B) = V^{(n)}(0, s)$ as the discrete-time approximation for the option price.

Algorithm. Backward induction in the RB tree

1. Set $V^{(n)}(T, S^{(n)}(n)) = f(S^{(n)}(n))$.
2. For $i = n-1, \dots, 0$ do

$$\begin{aligned} &V^{(n)}\left(i \cdot \Delta t, S^{(n)}(i)\right) \\ &\quad = \frac{1}{2}\left[V^{(n)}\left((i+1) \cdot \Delta t, uS^{(n)}(i)\right) + V^{(n)}\left((i+1) \cdot \Delta t, dS^{(n)}(i)\right)\right] \cdot e^{-r\Delta t}. \end{aligned}$$

3. Set $E^{(n)}(e^{-rT}B) = V^{(n)}(0, s)$ as the discrete-time approximation for the option price.

Apparently, due to the symmetry in probabilities, the RB model requires less operation counts for backward induction than the CRR model.

Note that for path-dependent options, it depends on the specific payoff functional whether there exist suitable modifications of the above algorithm. In particular, the algorithm can easily be adapted to the valuation of American options. Due to the widespread use of American options, this is an important advantage of the binomial method compared to alternative valuation techniques such as e.g. Monte Carlo methods. American options can be exercised at any time between the purchasing date and the expiration date T. In the Black-Scholes setting, this small conceptual difference causes a big difference in pricing because the optimal exercise date is not known on the date of purchase. Rather, it depends on the random evolution of the stock price process and is therefore itself random (mathematically, it is a stopping time with respect to the filtration generated by S). In contrast to the continuous-time American valuation problem, the American valuation problem can always (i.e. for any payoff function) be solved explicitly in the binomial model. Indeed, the main modification to the above backward induction algorithm is that for each node of the tree, the exercise value (i.e. the *intrinsic value* of the option) has to be compared to the value obtained by holding the option at least until the next time period and exercising it optimally afterwards. Let us illustrate binomial pricing of American options for the RB tree:

Algorithm. Backward induction for American options in the RB tree

1. Set $V^{(n)}(T, S^{(n)}(n)) = f(S^{(n)}(n))$.
2. For $i = n-1, \ldots, 0$ do

$$\begin{aligned}\tilde{V}^{(n)} &\left(i \cdot \Delta t, S^{(n)}(i)\right) \\ &= \frac{1}{2}\left[V^{(n)}\left((i+1)\cdot \Delta t, uS^{(n)}(i)\right) + V^{(n)}\left((i+1)\cdot \Delta t, dS^{(n)}(i)\right)\right]\cdot e^{-r\Delta t}\end{aligned}$$

   and set

$$V^{(n)}\left(i \cdot \Delta t, S^{(n)}(i)\right) = \max\left\{\tilde{V}^{(n)}\left(i \cdot \Delta t, S^{(n)}(i)\right), f\left(S^{(n)}(i)\right)\right\}.$$

3. Set $E^{(n)}(e^{-rT}B) = V^{(n)}(0, s)$ as the discrete-time approximation for the price of an American option with final payment f.

Note. Due to the simplified dynamics of binomial models (finite state space and discrete-time observations), binomial approximations to Black-Scholes option prices can be obtained by an easy and efficient backward induction algorithm. This is useful in practical applications if an analytic pricing formula is not known in the Black-Scholes setting. In particular, as seen above, the tree algorithm can easily be modified to the valuation of American options.

Although the binomial method is based on an efficient backward induction algorithm, it suffers from several drawbacks in practical applications. First, if the payoff function is discontinuous, the Berry-Esséen inequality on the rate of convergence of binomial price estimates is in general tight; i.e. convergence is no faster than $1/\sqrt{n}$. Second, for many types of options convergence is not smooth, but oscillatory: we observe low-frequency shrinking accompanied by high-frequency oscillations. Consequently, choosing a smaller grid size does not necessarily provide a better option price estimate, and extrapolation methods can typically not be applied. Well-known examples of irregular convergence behavior are the so-called *sawtooth effect* and the *even-odd problem*: Binomial price estimates obtained from the conventional methods described above often exhibit a sawtooth pattern. That is, if the grid size is increased ($n \to n+2$), the discretization error in the corresponding binomial option prices decreases to a negligible size. However, if the step size is further increased, the error rises abruptly. This is again followed by a period of decreasing errors. Figure 1 illustrates the sawtooth pattern. The sawtooth effect was first observed for

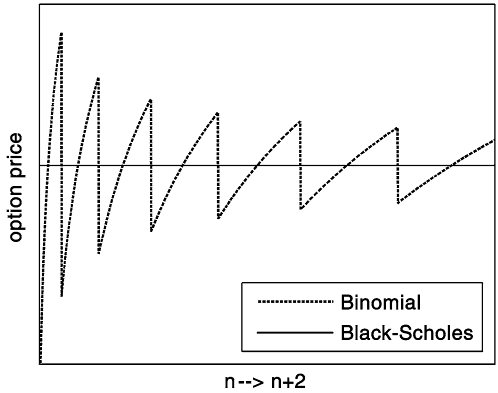

Fig. 1 The sawtooth effect

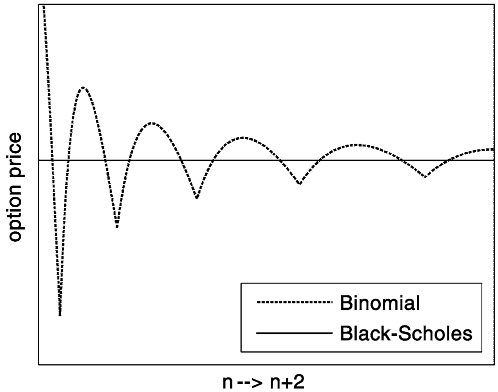

Fig. 2 Scallops

barrier options (i.e. options for which the right to exercise either originates or expires on certain regions of the path space of S) by Boyle and Lau (1994); yet it can also be present for other types of options. The price estimates obtained from conventional tree methods (for $n \to n+2$) also often converge in form of scallops. This is illustrated in Fig. 2. In this survey, we do not wish to explain where the different patterns come from, but let us stress that the convergence pattern observed depends both on the valuation problem under consideration and on the parameter specification of the tree method chosen. In addition to the irregular convergence behavior observed along $n \to n+2$, the binomial price estimates typically exhibit micro oscillations between even and odd values of n; the latter aspect is often referred to as the even-odd effect. The micro oscillations are superimposed on the macro oscillations considered previously; i.e. they are superimposed on the irregular convergence behavior along the even integers (the sawtooth pattern, scallops, etc.) and on the irregular convergence behavior along the odd integers.

There is a vast number of articles on controlling the discretization error, amongst which are Leisen and Reimer (1996), Leisen (1998), Tian (1999) and Chang and Palmer (2007). Leisen and Reimer use an odd number of periods with the tree centered around the strike value of interest. Leisen uses an even number of periods with the central node placed exactly at the strike. For the model suggested by Tian (Tian tree) and by Chang and Palmer (CP tree), the nodes in the tree are moved only a small distance so that the strike falls onto a neighboring node or onto the geometric average of the two neighboring nodes, respectively.

Example 1 (The Tian Tree). Let $K \in \mathbb{R}$ be arbitrary. For binomial valuation of call and put options, the point K will be the strike value. The basic idea behind the Tian model is that for any number of periods n, the terminal distribution of the CRR tree is modified so that it admits a realization placed exactly at the point K. To be precise, for each $n \in \mathbb{N}$, there is some integer $l(n)$ for which $K \in (s_n^{(n)}(l(n)-1), s_n^{(n)}(l(n))]$, where $s_n^{(N)}(l(n)-1)$ and $s_n^{(N)}(l(N))$ are adjacent terminal nodes in the CRR tree. Given the sequence $(l(n))_n$, we define a sequence $(t(n))_n$ with

$$t(n) := \frac{\ln(K/s_0) - (2l(n)-n)\sigma\sqrt{T/n}}{T}.$$

While the log-tree suggested by Cox, Ross and Rubinstein is symmetric around the starting value, the log-tree is now tilted by $t(n)\Delta t$. As a result, the strike value always coincides with a terminal node (compare Chang and Palmer 2007 for details[1]). Note that the tilt $t(n)$ depends on the number of periods n. It can be verified that the tilt is sufficiently small to maintain weak convergence to the stock price process S. As shown in Chang and Palmer (2007), the resulting binomial tree shows an improved convergence behavior compared to the CRR tree: For cash-or-nothing options (options with a piecewise constant payoff), the estimates still converge in order $1/\sqrt{n}$, but convergence is smooth, so extrapolation methods can be applied. For plain-vanilla options (options with a piecewise linear payoff), the estimates converge in order $1/n$. In contrast to the CRR tree, the coefficient of the leading error term is again constant.

Example 2 (The CP Tree). The CP model is such that for any number of periods n, the strike K is optimally located between two adjacent terminal nodes. The geometry of the CRR tree implies that the strike value is optimally placed if it is set at the geometric average of two adjacent nodes (compare Chang and Palmer 2007 for details). This can again be achieved by defining an appropriate sequence of tilt parameters: Let $(l(n))_n$ be defined as above. Then, the appropriate sequence of tilt parameters is given by $(\tilde{t}(n))_n$ with

$$\tilde{t}(n) = \frac{\ln(x/s_0) - (2l(n)-n-1)\sigma\sqrt{T/n}}{T}.$$

[1] In the original article by Tian, the improved convergence behavior is illustrated by numerical examples. Theoretical results are given in Chang and Palmer (2007).

As for the Tian model, the resulting tree remains close enough to the CRR tree to ensure weak convergence. The convergence behavior of the CP model is further improved compared to the Tian model. For the latter, the probability to end up in the money (i.e. the likelihood that your bet on the stock price movement is correct) is consistently under- or overestimated. The CP model takes account of this problem. As a result, the rate of convergence for cash-or-nothing options is improved to $1/n$ (without extrapolation).

A conceptually different approach to improve the convergence behavior of the discretization error of binomial trees can be found in Rogers and Stapleton (1998). They fix some $\Delta x > 0$ and view the diffusion only at the discrete set of times at which it has moved Δx from where it was last observed. This approximation technique results in a random walk that approximates the diffusion uniformly closely, so that the convergence behavior can be improved without an explicit re-location of nodes in the tree. However, it leads to a pathwise binomial tree with a random number of periods.

Remark (Trinomial Trees). As the completeness of the binomial market is irrelevant for numerical valuation of Black-Scholes option prices, the approximation by tree methods is not limited to binomial models; that is, models that exhibit two-state movements. In fact, the Black-Scholes option price can be approximated by any k-nomial tree (i.e. a tree for which each node has the same number k of successor nodes) provided the tree model satisfies the moment matching conditions required to apply Donsker's Theorem. Furthermore, if the corresponding tree is *re-combining*, a backward induction algorithm similar to that described for binomial trees can be used to efficiently compute discounted expectations in the k-nomial model. In a re-combining tree, paths with the same number of up- and down-movements end at the same node independently of the order in which the up- and down-movements have occurred. Of course, if the tree model allows for additional states, the computational effort required for backward induction increases compared to binomial trees. However, the application of multinomial models provides some additional free parameters because there are two moment matching conditions only—independently of the number of states in the discrete-time model. Consequently, recombining trinomial trees (i.e. trees for which each node has three successors) are sometimes used in practical applications to approximate the price of path-dependent options (such as barrier options) as they are more flexible than their binomial counterparts.

Note. Trinomial trees can be adapted to complex valuation problems—this can lead to an improved convergence behavior. However, trinomial option valuation is more costly than binomial pricing. Alternatively, the convergence behavior can be improved without increasing computational effort by applying advanced binomial models. The implementation of advanced binomial models can be more involved.

4 Multi-Asset Option Valuation and Binomial Trees

4.1 Standard Binomial Methods for Multi-Asset Options

Compared to the single-asset case, setting up multi-dimensional binomial trees is more complicated because the entire correlation structure between the underlying (log-)asset prices in the Black-Scholes setting have to be taken into account. More precisely, if we use a binomial approximation to the multi-asset Black-Scholes model given by

$$dS_i(t) = S_i(t)\left(r dt + \sigma_i dW_i\right), \qquad \mathrm{Corr}\left(\frac{dS_i}{S_i}, \frac{dS_j}{S_j}\right) = \rho_{ij} dt, \quad i, j = 1, \ldots, m, \tag{3}$$

the correlations between the log-prices have to be matched in addition to the previous conditions on the expectation and the variance of the one-period log-returns. Consequently, the construction of multi-dimensional binomial trees becomes the more involved, the more assets are traded in the market. Furthermore, the additional moment matching conditions can lead to negative jump probabilities in the corresponding binomial model, so that the model is no longer well-defined. In particular, the above argument on weak convergence cannot be used anymore.

We define an m-dimensional n-period binomial tree via

$$S_i^{(n)}(k) = S_i^{(n)}((k-1))\, e^{\alpha_i \Delta t + \beta_i \sqrt{\Delta t} Z_{k,i}^{(n)}}, \quad k = 1, \ldots, n,\ i = 1, \ldots, m,$$

where as before each period has length $\Delta t = \frac{T}{n}$ and $Z_{k,i}^{(n)}$ are random variables taking values in $\{-1, 1\}$. We choose the constants α_i, β_i and the jump probabilities (defining the distribution of the random variables $Z_{k,i}^{(n)}$) such that

- the random vectors $(Z_{k,1}^{(n)}, \ldots, Z_{k,m}^{(n)})$, $k = 1, \ldots, n$, are i.i.d. for fixed n,
- the first two moments of the one-period log-returns in the Black-Scholes model coincide (at least) asymptotically with those in the tree; in particular, the covariances of the random variables $Z_{k,i}^{(n)}$ and $Z_{k,j}^{(n)}$ satisfy

$$\beta_i \beta_j \,\mathrm{Cov}\left(Z_{k,i}^{(n)}, Z_{k,j}^{(n)}\right) \stackrel{n\to\infty}{\longrightarrow} \rho_{ij}\sigma_i\sigma_j, \quad 1 \le i, j \le m.$$

Let us illustrate standard multi-asset binomial trees with multi-dimensional generalizations of the (one-dimensional) CRR tree and the (one-dimensional) RB tree:

Example 3 (The BEG Tree). The BEG tree as introduced in Boyle et al. (1989) is the m-dimensional generalization of the one-dimensional CRR tree; that is

$$S_i^{(n)}(k) = S_i^{(n)}((k-1))\, e^{\sigma_i \sqrt{\Delta t} Z_{k,i}^{(n)}}, \quad k = 1, \ldots, n,\ i = 1, \ldots, m,$$

where for each period k, the distribution of the random vector $(Z_{k,1}^{(n)}, \ldots, Z_{k,m}^{(n)})$ is defined via the set of jump probabilities

$$p_{BEG}^{(n)}(l) = \frac{1}{2^m}\left(1 + \sum_{j=1}^{m}\sum_{i=1}^{j-1}\delta_{ij}(l)\rho_{ij} + \sqrt{\Delta t}\sum_{i=1}^{m}\delta_i(l)\frac{r - \frac{1}{2}\sigma_i^2}{\sigma_i}\right), \quad 1 \le l \le 2^m,$$

with

$$\delta_i(l) = \begin{cases} 1 & \text{if } Z_{k,i}^{(n)} = 1, \\ -1 & \text{if } Z_{k,i}^{(n)} = -1, \end{cases} \qquad \delta_{ij}(l) = \begin{cases} 1 & \text{if } Z_{k,i}^{(n)} = Z_{k,j}^{(n)}, \\ -1 & \text{if } Z_{k,i}^{(n)} \neq Z_{k,j}^{(n)}. \end{cases} \tag{4}$$

As for the CRR tree, the log-prices in the BEG-tree are symmetric around the starting value. However, the probabilities depend in a complicated way on both the drift and the covariance structure of the continuous-time model.

Example 4 (The m-Dimensional RB Tree). The m-dimensional Rendleman-Barrter tree is described in Amin (1991) and in Korn and Müller (2009). It is given by

$$S_i^{(n)}(k) = S_i^{(n)}((k-1))\, e^{\left(r - \frac{1}{2}\sigma_i^2\right)\Delta t + \sigma_i\sqrt{\Delta t}Z_{k,i}^{(n)}}, \quad k = 1, \ldots, n, \; i = 1, \ldots, m,$$

where for each period k, the distribution of the random vector $(Z_{k,1}^{(n)}, \ldots, Z_{k,m}^{(n)})$ is defined via the set of jump probabilities

$$p_{RB}^{(n)}(l) = \frac{1}{2^m}\left(1 + \sum_{j=1}^{m}\sum_{i=1}^{j-1}\delta_{ij}(l)\rho_{ij}\right), \quad 1 \le l \le 2^m,$$

with $\delta_{ij}(.)$ given as in (4). While the log-prices in the RB-tree are non-symmetric, the probabilities depend only on the covariance structure of the continuous-time model. In particular, in contrast to the BEG tree, the jump probabilities depend neither on the number of periods n nor on the drift.

Let us emphasize that for both models, the jump probabilities are not necessarily non-negative! In contrast to the common belief, this problem cannot always be fixed by choosing a sufficiently large number of periods n (see Korn and Müller 2009 for an explicit example).

Remark (Incompleteness of Multi-Dimensional Binomial Trees). There is no analogue to Theorem 1 for multi-dimensional binomial trees. In particular, multi-dimensional binomial markets are incomplete for dimensions $m > 1$. This implies that there is in general no unique price for a given option payment. However, the incompleteness of the multi-dimensional binomial model is irrelevant for the application of the corresponding binomial model to numerical valuation of options in the corresponding (complete) multi-dimensional Black-Scholes market: Provided

- the model is well-defined, and
- the first two moments of the one-period log-returns are (asymptotically) matched,

the option price in the Black-Scholes model can be approximated by computing the expected discounted option payments in the multi-dimensional binomial models (with respect to the measure induced by the distribution of the random variables

$Z_{k,j}^{(n)}$). As for the one-dimensional setting, this procedure is justified by weak convergence arguments based on Donsker's Theorem. For an approximation of the multi-dimensional Black-Scholes model by a complete multinomial model, we refer to He (1990).

For the standard multi-dimensional trees considered above, the corresponding algorithm for binomial option pricing is conceptually the same as for the one-dimensional case. First, we assign possible payoff scenarios to the terminal nodes. Afterwards, we step backwards through the tree by computing the value at each node as the weighted average of the values assigned to its successor nodes. However, the number of terminal nodes grows exponentially in the number of traded assets (this effect is often referred to as the *curse of dimensionality*). Consequently, for high-dimensional options (i.e. options whose payments depend on a large number of assets), standard multi-dimensional trees are currently not practically useful. This is an inherent drawback of the binomial approach as a method based on the discretization of the underlying assets. However, up to dimension four, let us say, standard multi-dimensional trees can lead to results that are perfectly competitive and often superior to those obtained by Monte Carlo methods. Nevertheless, as inherited by their one-dimensional counterparts, standard multi-dimensional trees often exhibit an irregular convergence behavior.

Note. In principle, we can obtain multi-dimensional variants of the conventional one-dimensional binomial trees considered above. However, the corresponding trees are not well-defined for every parameter setting of the corresponding continuous-time model. Provided that the tree is well-defined and that it converges weakly to the multi-dimensional Black-Scholes stock price model, it can be applied to numerical valuation of options in the limiting Black-Scholes market. This yields an easy and efficient method to approximate the price of a multi-asset option, yet the corresponding tree model still suffers from several drawbacks with respect to practical applications.

In the following, we suggest orthogonal trees as an alternative to standard multi-dimensional trees.

4.2 Valuing Multi-Asset Options by Orthogonal Trees

The complications observed above motivate searching for alternative approaches to multi-dimensional trees that ensure well-defined jump probabilities and easy moment matching. In this context, Korn and Müller (2009) introduced orthogonal trees, which are based on a general decoupling method for multi-asset Black-Scholes settings. The model we suggest contains the two-dimensional example of Hull (2006) as a special case.

The construction of orthogonal trees consists of two steps: In contrast to standard multi-dimensional tree methods, we first decouple the components of the log-stock price process $(\ln(S_1), \ldots, \ln(S_m))$. Afterwards, we approximate the decoupled process by an m-dimensional tree defined as the product of appropriate one-dimensional trees. Of course, in order to apply the resulting tree to numerical option valuation, we have to apply a backtransformation to any time-layer of nodes that contributes to the option payment under consideration.

To explain the above procedure in detail, we consider the multi-asset Black-Scholes type market with m stocks following the price dynamics (3). The decoupling method is based on a decomposition of the volatility matrix Σ of the log-prices $(\ln(S_1(t)), \ldots, \ln(S_m(t)))$ via

$$\Sigma = GDG^T \tag{5}$$

with D a diagonal matrix. The above decomposition allows to introduce the transformed ("decoupled") log-price process Y defined by

$$Y(t) := G^{-1}\left(\ln\left(S_1(t)\right), \ldots, \ln\left(S_m(t)\right)\right)^T.$$

Note that the process $Y = (Y_1, \ldots, Y_m)$ follows the dynamics

$$\begin{aligned} dY_j(t) &= \mu_j dt + \sqrt{d_{jj}}d\bar{W}_j(t), \qquad \mu = G^{-1}\left(r\underline{1} - \frac{1}{2}\underline{\sigma}^2\right), \\ \underline{\sigma}^2 &= \left(\sigma_1^2, \ldots, \sigma_m^2\right)^T, \end{aligned} \tag{6}$$

where $\bar{W}(t)$ is an m-dimensional Brownian motion (see Korn and Müller 2009 for details). In particular, the component processes are independent. Consequently, the transformed process Y can be approximated by a product of (independent) one-dimensional trees; one for each component process Y_j. This implies that we have to match only the mean and variance of one-period log-returns for each component process. There is no need for correlation matching! In particular, it is easy to obtain well-defined transition probabilities.

The decoupling approach allows for the following choices:

- Which one-dimensional tree(s) should be used to approximate the components?
- Which matrix decomposition should we choose in (5)?

In fact, each component process Y_j can be approximated by an arbitrary one-dimensional (factorial) tree. As discussed previously, the one-dimensional RB tree leads to a backward induction algorithm that is cheaper than that obtained by the CRR tree. Consequently, we suggest to approximate each component Y_j by an appropriate one-dimensional RB tree. In particular, this choice implies that the resulting m-dimensional tree is always well-defined (independently of both the parameter setting and the number n of periods). To answer the second question, note that there is an infinite number of decompositions solving (5). In particular, the Cholesky decomposition and the spectral decomposition can be applied. However, as shown by numerical and theoretical considerations, it is typically more favorable to choose the spectral decomposition[2] (compare Korn and Müller 2009 for details). Applying the decoupling approach with the above choices results in the following basic steps for numerical valuation of path-independent options with payoff $B = f(S_1(T), \ldots, S_m(T))$:

Numerical Valuation of Path-Independent Options via Orthogonal RB Trees

1. Compute the spectral decomposition $\Sigma = GDG^T$ of the covariance matrix of the log-stock prices.
2. Introduce the process $Y(t) := G^T(\ln(S_1(t)), \ldots, \ln(S_m(t)))^T$ following the dynamics (6).
3. Approximate each component process Y_j by a one-dimensional RB tree which matches the mean and the variance of the one-period log-returns. This results in an m-dimensional discrete process $Y^{(n)}$ following the dynamics

$$Y^{(n)}(k+1) = \begin{pmatrix} Y_1^{(n)}(k) + \mu_1 \Delta t + Z_{k+1,1}^{(n)} \sqrt{d_{11}} \sqrt{\Delta t} \\ \vdots \\ Y_m^{(n)}(k) + \mu_m \Delta t + Z_{k+1,m}^{(n)} \sqrt{d_{mm}} \sqrt{\Delta t} \end{pmatrix},$$
$$Y^{(n)}(0) = Y(0),$$

   where $(Z_{k+1,1}^{(n)}, \ldots, Z_{k+1,m}^{(n)})$ is a random vector of independent components, each attaining the two values $+1$ and -1 with probability $\frac{1}{2}$. As above, we require that the random vectors $(Z_{k,1}^{(n)}, \ldots, Z_{k,m}^{(n)})$, $k = 1, \ldots, n$, are i.i.d. Further, d_{ii} are the eigenvalues of the covariance matrix Σ. By moment matching the drift vector $(\mu_1, \ldots, \mu_m)$ and the distribution of $(Z_{k+1,1}, \ldots, Z_{k+m,1})$ are determined in such a way that the process $Y^{(n)}$ approximates the decoupled process Y.
4. Apply the backtransformation of the decoupling rule to the terminal nodes in the tree associated with $Y^{(n)}$, i.e. with $h(\underline{x}) := (e^{G_1 \underline{x}}, \ldots, e^{G_m \underline{x}})$ (where G_i denotes the i th row of G) set

$$S^{(n)}(n) := h\left(Y^{(n)}(n)\right).$$

   Starting from the transformed terminal nodes (i.e. from the realizations of the random variable $S^{(n)}(n)$), we obtain an approximation to the Black-Scholes option price using the standard backward induction algorithm.

It remains to answer the question whether we can theoretically justify the above procedure to approximate Black-Scholes option prices via the decoupling approach. In fact, one can show by weak convergence arguments that this is ensured by continuity of the backtransformation map.

[2] The triangular structure of the Cholesky decomposition is favorable when additional assets enter the market. This issue will not be addressed in this survey.

Numerical Performance of Orthogonal RB Trees

For path-independent options, the above valuation algorithm is cheaper than the conventional multi-dimensional tree methods considered previously. This is due to the fact that under the above choice of the embedded one-dimensional trees, each path in the m-dimensional tree is equally likely. However, for path-dependent options, it does not suffice to apply the backtransformation to the terminal nodes only. Rather, the backtransformation has to be applied to every time layer in the tree that contributes to the option payments. In particular, for the valuation of American options, all time layers have to be transformed. This means that we have to transform the entire tree associated with $Y^{(n)}$ into a "valuation tree" associated with the discrete process $S^{(n)}$ defined by

$$S^{(n)}(k) := h\left(Y^{(n)}(k)\right), \quad k = 0, \dots, n.$$

Due to the additional computational effort required for backtransformation, numerical valuation of path-dependent and American options is typically more expensive for orthogonal trees than for conventional multi-dimensional trees. However, in case that the latter are not well-defined, the decoupling approach at least justifies binomial option valuation. Furthermore, the decoupling approach often leads to a more regular convergence behavior than that observed for conventional multi-dimensional methods. This is a consequence of the fact that by applying the backtransformation to the nodes in the tree, the nodes are dislocated in an irregular (i.e. non-linear) way. As a result, the probability mass gets smeared in relation to fixed strike values or barrier levels. The benefits due to the more regular convergence behavior often overcompensate the additional computational effort required for backtransformation. In particular, the sawtooth effect can vanish completely, so that the order of convergence can be improved by applying extrapolation methods (for a detailed performance analysis see Korn and Müller 2009).

In addition to the above advantages, the decoupling approach is perfectly suited to cut down high-dimensional valuation problems to the "important dimensions". To explain this, note that the dynamics of stock markets can often be explained by a relatively small number of risk factors that is less than the number of traded assets. In such a market, it seems reasonable to value an option by a tree whose dimension is lower than the number of underlyings. The above algorithm is particularly suited to that purpose because it incorporates a principal component analysis in an implicit way. In particular, it considers the underlying risk factors (rather than the traded stocks) as the important ingredients. Consequently, if the number of relevant risk factors is small, the decoupling approach can give a fast first guess on high-dimensional valuation problems by considering the non-relevant risk factors as deterministic (compare Korn and Müller 2009).

Note. We suggest orthogonal trees as an alternative to standard multi-dimensional trees. On the one hand, the decoupling approach keeps the tree structure which results in an efficient backward induction algorithm; on the other hand, it is no longer based on a random walk approximation to the correlated asset price processes, which leads to advantages in practical applications. In particular, the above orthogonal tree procedure is always well-defined, and it can be combined with a principal component analysis, which leads to model reduction. This will allow for a fast first guess on the solution of high-dimensional valuation problems.

5 Conclusion and Outlook

Despite their conceptual simplicity, binomial trees can offer an efficient numerical method to approximate Black-Scholes option prices. This is in particular true for American options. However, multi-dimensional option valuation by binomial trees suffers from the inherent drawback that it is currently not of practical use for high-dimensional problems. Furthermore, as conventional binomial trees often lead to an irregular convergence behavior, controlling the discretization error is important in practical applications.

Binomial methods can also be applied to approximate option prices in continuous-time stock price models other than the Black-Scholes model. There is still ongoing research on the application of the binomial method to stock price models following non-Black-Scholes-type dynamics.

Further, the irregular convergence behavior of binomial methods remains a field of intensive study. Previous research often concentrates on a particular type of options (such as barrier options) and thus leads to highly specialized approaches. An exception is the orthogonal tree method of Korn and Müller (2009) which seems to exhibit a more smooth convergence behavior for the popular types of exotic options.

As shown above, the orthogonal tree procedure can be cut down to important risk factors. We suggest to analyse this issue for options on prominent indices. Further, one could think of alternative approaches to deal with the curse of dimensionality.

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