Optimal Market Making in Discrete Time

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1 Trading order book (TOB)

We consider a financial market composed by a single asset. We start by defining the most spread mechanism to make exchanges on financial markets: the trading order book. For that we describe the rules defining it.

- Buyers/Sellers express their intent to trade by submitting bids/asks.
- These are Limit Orders (LO) with a price P and size N.
- Buy LO (P, N) states willingness to buy N shares at a price cheaper than (or equal to) P.
- Sell LO (P, N) states willingness to sell N shares at a price more expensive than (or equal) P.
- Trading Order Book aggregates order sizes for each unique price.
- So we can represent with two sorted lists of $(Price, Size)$ pairs

$$\begin{aligned} Bids &= \{(P_i^{(b)}, N_i^{(b)})\ ,\ 1 \leq i \leq m\} \text{ with } P_j^{(b)} > P_i^{(b)} \text{ for } i < j\ , \\ Asks &= \{(P_i^{(a)}, N_i^{(a)})\ ,\ 1 \leq i \leq n\} \text{ with } P_j^{(a)} < P_i^{(a)} \text{ for } i < j\ , \end{aligned}$$

- We call $P_1^{(b)}$, $P_1^{(a)}$ and $\frac{P_1^{(b)}+P_1^{(a)}}{2}$ as Bid, Ask and Mid respectively.

- We call $P_1^{(b)} - P_1^{(a)}$ and $P_n^{(b)} - P_m^{(a)}$ as Spread and Market Depth.

The next picture summarizes the definitions presented above.

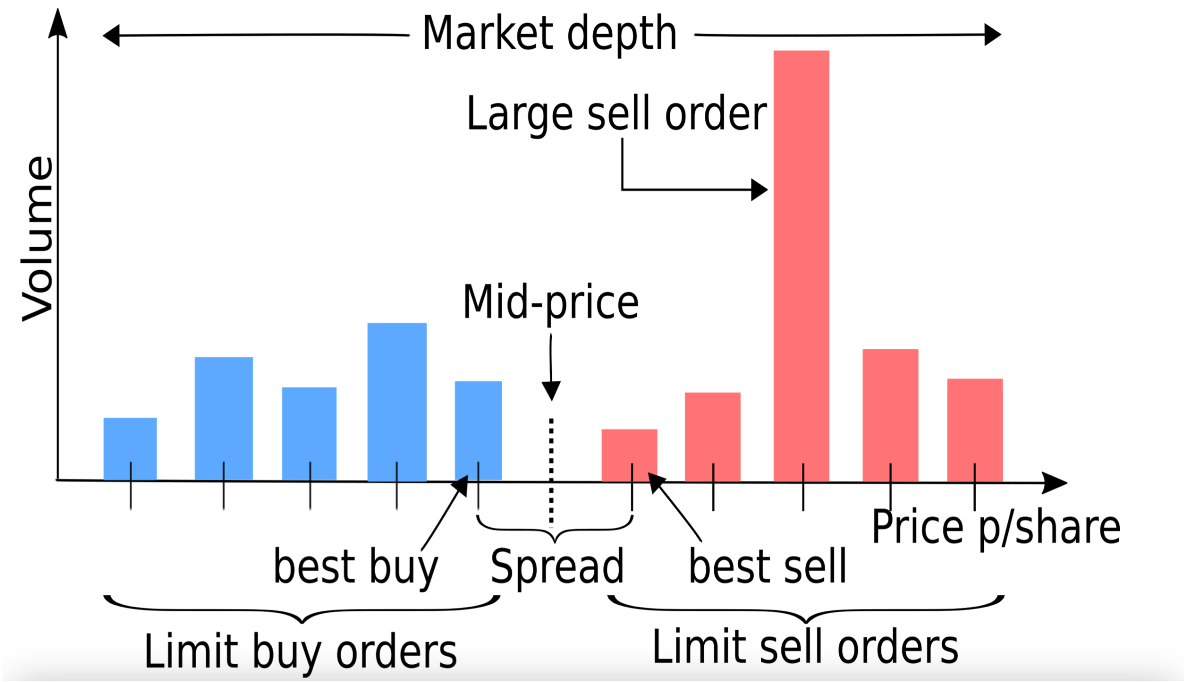

Figure 1: The Limit Order Book

A Market Order (MO) states intent to buy/sell N shares at the best possible price(s) available on the TOB at the time of MO submission. We now describe the TOB activity.

- A new Sell LO (P, N) potentially removes best bid prices on the TOB

$$Removal \quad : \quad (P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)})) \text{ for } i \text{ s.t. } P_i^{(b)} \geq P \ .$$

- After this removal, it adds the following to the asks side of the TOB

$$\left(P, \max(0, N - \sum_{i:P_i^{(b)} \geq P} N_i^{(b)})\right) \ .$$

- A new Buy MO operates analogously (on the other side of the TOB).
- A Sell Market Order N will remove the best bid prices on the TOB

$$\text{Removal of orders } i \quad \text{s.t.} \quad N_i^{(b)} \leq N - \sum_{j=1}^{i} N_j^{(b)})) \text{ and } P_i^{(b)} \geq P \ .$$

- A Buy Market Order (P, N) will remove the best ask prices on the TOB

$$\text{Removal of orders } i \quad \text{s.t.} \quad N_i^{(a)} \leq N - \sum_{j=1}^{i} N_j^{(a)})) \text{ and } P_i^{(a)} \leq P \ .$$

Modeling TOB Dynamics involves predicting arrival of MOs and LOs. Market-makers are liquidity providers (providers of Buy and Sell LOs). Other market participants are typically liquidity takers (MOs)

But there are also other market participants that trade with LOs Complex interplay between market-makers $\&$ other mkt participants Hence, TOB Dynamics tend to be quite complex We view the TOB from the perspective of a single market-maker who aims to gain with Buy/Sell LOs of appropriate width/size By anticipating TOB Dynamics $\&$ dynamically adjusting Buy/Sell LOs Goal is to maximize Utility of Gains at the end of a suitable horizon.

If Buy/Sell LOs are too narrow, more frequent but small gains. If Buy/Sell LOs are too wide, less frequent but large gains.

Market-maker also needs to manage potential unfavorable inventory (long or short) buildup and consequent unfavorable liquidation.

2 Optimization problem for the market-maker

We simplify the setting for ease of exposition. We first assume finite time steps indexed by $t = 0, 1, \ldots, T$. We next fix the following notations.

- W_t denotes the market-maker's trading PnL at time t,
- $I_t \in \mathbb{Z}$ denotes the market-maker's inventory of shares at time t (with $I_0 = 0$),
- S_t denotes the TOB Mid Price at time t.

- $P_t^{(b)}, N_t^{(b)}$ are market maker's Bid Price, Bid Size at time t,
- $P_t^{(a)}, N_t^{(a)}$ are market-maker's Ask Price, Ask Size at time t,
- $\delta_t^{(b)} = S_t - P_t^{(b)}$ denotes the Bid Spread and $\delta_t^{(a)} = P_t^{(a)} - S_t$ dentes the Ask Spread,
- the random variable $Y_t^{(b)} \in \mathbb{N}$ denotes bid-shares "hit" up to time t,
- the random variable $Y_t^{(a)} \in \mathbb{N}$ denotes ask-shares "lifted" up to time t.

We assume market-maker can add or remove bids/asks costlessly. The dynamics of the PnL is then given by

$$W_{t+1} = W_t + P_t^{(a)}(Y_{t+1}^{(a)} - Y_t^{(a)}) - P_t^{(b)}(Y_{t+1}^{(b)} - Y_t^{(b)})$$

for $t = 0, \dots, T-1$. We also assume that the preferences of the market-marker are given by a utility function U. The goal of the market-maker is to maximize the expected terminal utility

$$\mathbb{E}\Big[U\Big(W_T + I_T S_T\Big)\Big]$$

over the strategies $(P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)})_{t=0}^{T-1}$.

3 Controlled Markov process

Denote by $(X_t)_{t=0}^T$ the related state process. It is then given by

$$X_t = (S_t, W_t, I_t)$$

for $t = 0, \dots, T$. We next assume that

$$\begin{aligned} Y_{t+1}^{(a)} - Y_t^{(a)} &\sim Bernoulli\big(f^{(a)}(\delta_t^{(a)})\big) \\ Y_{t+1}^{(b)} - Y_t^{(b)} &\sim Bernoulli\big(f^{(b)}(\delta_t^{(b)})\big) \end{aligned}$$

for $t = 0, \dots, T$ with $f^{(a)}$ and $f^{(b)}$ are two decreasing functions. We also assume that $Y_{t+1}^{(a)} - Y_t^{(a)}$ and $Y_{t+1}^{(b)} - Y_t^{(b)}$ are independent of the past up to time t. Since Bernoulli random variables take at most value 1, $N_t^{(b)}$ and $N_t^{(a)}$ can be assumed to be equal to 1.

This simplifies the action at time t to be just the pair $(\delta_t(b), \delta_t^{(a)})$ since the dynamics of state X can be expressed through this pair:

$$\begin{aligned} W_{t+1} &= W_t + (S_t + \delta_t^{(a)})(Y_{t+1}^{(a)} - Y_t^{(a)}) - (S_t - \delta_t^{(b)})(Y_{t+1}^{(b)} - Y_t^{(b)}) \\ I_{t+1} &= I_t + (Y_{t+1}^{(a)} - Y_t^{(a)}) - (Y_{t+1}^{(b)} - Y_t^{(b)}) \end{aligned}$$

for $t = 0, \dots, T-1$. We assume that the mid price follows a log-normal dynamics given by

$$S_{t+1} = S_t(1 + e^{\sigma Z_t})$$

for $t = 0, \dots, T-1$, where $Z_0, \dots, Z_{T-1}$ are IID random variables such that $\mathbb{P}(Z_t = 1) = \mathbb{P}(Z_t = -1) = 1/2$.

The process $(X_t)_{t=0}^T$ is then a controlled Markov process with controlled transitions $(P_t)_{t=0}^T$ given by

$$\begin{aligned} P_t^d f(x) = \frac{1}{2} \sum_{p,\ell=0,1 r=\pm 1} & f\big(s(1+e^{r\sigma}), w + (s + d^{(a)})p - (s - d^{(b)})\ell, i + p - \ell\big) \\ & f^{(a)}(d^{(a)})^p (1 - f^{(a)}(d^{(a)}))^{1-p} f^{(b)}(d^{(b)})^\ell (1 - f^{(b)}(d^{(b)}))^{1-\ell} \end{aligned}$$

for $x = (s, w, i)$, $d = (d^{(a)}, d^{(b)})$ and f a measurable bounded (or positive) function.

The dynamic programming equations define the functions $J_0, \dots, J_T$ in a recursive way by

$$J_T(x) = U(w + is) , \quad x = (s, w, i) ,$$

and

$$J_t(x) = \max_d P_t^d J_{t+1}(x)$$

for $t = 0, \dots, T-1$. Then we have

$$J_0(x) = \sup_\delta \mathbb{E}\big[U(W_T + S_T I_T) | X_0 = x\big] .$$

4 The case of exponential utility

We suppose that the utility function U is given by

$$U(x) = -\exp(-\rho x) , \quad x \in \mathbb{R} .$$

In this case the functions $J_0, \dots, J_T$ take the form

$$J_t(s, w, i) \;\; = \;\; -\exp\big(-\rho\big(w + \theta_t(s, i)\big)\big)$$

with $\theta_T(s, i) = si$ and

$$\theta_t(s, i) \;\; = \;\; -\frac{1}{\rho}\ln\Big(-\sup_{(d^{(a)}, d^{(a)})} P_t^{(d^{(a)}, d^{(a)})}\Big(-e^{-\rho\theta_{t+1}(s,i)}\Big)\Big)\,.$$